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Problems in Arithmetic

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Contents

Foreword

The present book contains problems in arithmetic for pupils of the 5th and 6th forms. It includes all the arithmetic problems from the books “Problems in Arithmetic and Visual Geometry (5th Form)”, MCCME Publ., 2020 and “Problems in Arithmetic and Visual Geometry (6th Form)”, MCCME Publ., 2021, as well as 300 additional problems, mostly more difficult ones.

These problems as compared to the ones proposed in the standard textbook in order to truly understand the material are somewhat more difficult. Problems of this kind are sometimes called entertaining or ingenuity problems. Ingenuity can be used to find solutions that cannot be obtained by standard arguments covered in school. The problems are entertaining in the sense that pupils faced with such problems enjoy figuring them out. Of course, not all the problems are simultaneously entertaining and require ingenuity. Some problems in the book are easy, others more difficult.

Chapter 1

Natural Numbers

The decimal system

1.1. Find all the natural numbers that are 5 times greater than their last digit.

1.2. Find all the two-digit numbers that are 7 times greater than their last digit.

1.3. Find a two-digit number which is 5 times greater than the sum of its digits.

1.4. Find a two-digit number which is 3 times greater than the sum of its digits.

1.5. I bought a lottery ticket with five-digit number whose sum of digits turned out to be equal to the age of my neighbour. Having learned about this coincidence, my neighbour managed to determine uniquely the number of this ticket. Can you find this number, too?

1.6. How many natural numbers n are there such that among the numbers n and $n + 937$ only one is a three-digit number?

1.7. Find a four-digit number whose first digit indicates how many zeros there are in its decimal expression, the second one, how many ones, the third, how many twos, the fourth, how many threes.

1.8. Alex wrote down the numbers from 1 to 100, Mike erased some of them. Among the remaining numbers, 20 have a one in their notation, 19, have a two, while 30 have neither a one nor a two. How many numbers did Mike erase?

1.9. Using each of the digits 1, 2, 3, and 4 twice, write an eight-digit number in which there is one digit between the ones, two digits between the twos, three, between the threes, four, between the fours.

1.10. Give an example of a ten-digit number whose first digit indicates the number of ones in that number, the second digit indicates the number of twos, ..., the ninth, the number of nines, the tenth, the number of zeros.

1.11. Give an example of a ten-digit number whose first digit indicates the number of zeros in that number, the second digit indicates the number of ones, ..., the tenth, the number of nines.

1.12. Give an example of two ten-digit numbers each of which describes the number of digits of the other, i.e., the first digit of one of them is equal to the number of zeros of the other (and vice versa), the second digit, the number of ones (and vice versa), and so on.

Comparison of numbers

1.13. In the number 5041638, delete five digits so that the obtained number should be maximal.

1.14. a) Construct the largest five-digit number whose decimal expression consists of different odd digits.

b) Construct the largest five-digit number whose decimal expression consists of different even digits.

1.15. In a family, the eldest sibling has 4 brothers and 2 sisters, and the youngest, 3 brothers and 3 sisters. Is the eldest sibling a boy or a girl?

1.16. Nick and Kate are in the same class. There are twice as many boys than girls in that class. Nick has 7 more boy classmates than girl classmates. How many girl classmates does Kate have?

1.17. Tony was given a balance, and he started weighing his toys. His car was held in balance by one ball and two cubes, while the car with one cube was held in balance by two balls. How many cubes will balance the car?

1.18. On a lawn there are as many barefoot boys as there are girls wearing shoes. Are there more barefoot children or girls on the lawn?

1.19. If each girl is given one chocolate, and each boy, two, then there will be enough chocolates. If each boy is given one chocolate, and each girl, two, then there won't be enough chocolates to go around. Now if we want to give no chocolates to the girls, and three to each boy, will there be enough?

1.20. Five children stood in line holding 37 flags. There were 14 flags to the right of Tania, 32 flags to the right of Jacob, 20 flags to the right of Vera, 8 flags to the right of Maxim. How many flags did Dasha hold?

How many numbers?

1.21. How many different three-digit numbers are there in whose expression each of the digits 1, 2, and 3 appears only once?

1.22. How many different three-digit numbers can be constructed from the digits 0, 2, 5, if the digits are not repeated in the expression of the number?

1.23. How many different three-digit numbers can be constructed from the digits 0, 2, 5, 7, if the digits are not repeated in the expression of the number?

1.24. How many different four-digit numbers can be constructed from the digits 1, 2, 5, 7, if the digits are not repeated in the expression of the number?

Add or subtract

1.25. Doctor Dolittle gave 2006 magic pills to four sick animals. Rhinoceros got one pill more than Crocodile, Hippopotamus got one pill more than Rhinoceros, and Elephant got one more than Hippopotamus. How many pills did Elephant get?

1.26. Together, Mary, Alex, and Dolly have 11 balloons. Mary has two balloons less than Dolly, Alex has one more balloon than Dolly. How many balloons does Dolly have?

1.27. A boy has the same number of sisters and brothers, and one of his sisters has half as many brothers as sisters. How many boys and how many girls are there among these siblings?

1.28. Johnny had 27 (wrapped) sweets in two pockets. He took from the right pocket as many sweets as he had in his left pocket and put them in the left pocket. After that, there were 3 more sweets in his right pocket than in the left one. How many sweets were there in each of the two pockets initially?

Seven more problems

1.29. There are 5 brothers in a family. Each brother has one sister. How many siblings are there in this family?

1.30. Prove that among three natural numbers, one can always choose two whose sum is even.

1.31. Donkeys were grazing in a clearing. Several boys came up to them. "Let each one of us mount a donkey," suggested the eldest boy. But then two boys were left without donkeys. “Let’s try to mount in twos,” the eldest boy suggested again. Then two boys mounted each donkey, but this time one donkey was left without a rider. How many donkeys and how many boys were there in the clearing?

1.32. In January there were four Fridays and four Mondays. What day of the week was January 1st?

1.33. Two clocks began and stopped ringing at the same time. The first one chimes every 2 s., the second, every 3 s. There was a total of 13 chimes heard (simultaneous chimes are regarded as one). How much time elapsed between the first and last chime?

1.34. One day Nick said, “The day before yesterday I was 10, while next year I will turn 13”. Is that possible?

1.35. Arrange the numbers 14, 27, 36, 57, 178, 467, 590, 2345 in a circle so that each two neighbouring numbers have a common digit.

Chapter 2

Operations with Natural Numbers

Arithmetical operations

2.1. In an exercise for summing two numbers, the first summand is less than the sum by 2000, and the sum is greater than the second summand by 20. Reconstruct the exercise.

2.2. Compare the numbers $101\,101 \cdot 999$ and $101 \cdot 999\,999$.

2.3. How many times do you have to add the largest two-digit number to the largest one-digit number in order to obtain the largest three-digit number?

2.4. The sum of the minuend, the subtrahend, and the difference is 28. Find the minuend.

2.5. Help Johnny to reconstruct an exercise in division of two numbers if it is known that that the quotient is five times less than the dividend and seven times greater than the divisor.

2.6. Represent the number 100 000 as a product of two natural numbers whose decimal expressions contain no zeros.

2.7. Among the numbers 21, 19, 30, 25, 3, 12, 9, 15, 6, 27 choose three numbers that sum to 50.

2.8. a) Prove that if one adds a 5-digit number and the number expressed via the same digits, but in inverse order, then at least one of the digits of the resulting sum will be even.

b) Prove the same for 17-digit numbers.

2.9. a) If one adds a three-digit number to a number expressed via the same digits but in inverse order, it is true that at least one digit of the sum will be necessarily even?

b) Same question for 16-digit numbers.

Problems about parts

2.10. Half of half of a number is equal to 1/2. What is that number?

2.11. Puss-in-Boots caught several pikes. He caught four plus half his entire catch. How many pikes did Puss-in-Boots catch?

2.12. A brick weighs 2 kilograms plus one third of its weight. How much does it weigh?

2.13. What part of a day (24 hours) has passed, if it constitutes half of the remaining part of the day?

2.14. Having travelled one third of his route, a train passenger fell asleep. When he woke up, he still had to travel half of the route that remained at the moment when he fell asleep. What part of the route was covered by the train as the passenger slept?

2.15. A merchant inadvertently mixed candies of the first quality (3 rubles a pound) and of the second quality (2 rubles a pound). At what price should he sell that mixture to obtain the same income if the total initial price of all the candies of the first quality is equal to the total initial price of all the candies of the second quality?

2.16. There are 80 pencils in three boxes. In the first box, there are 4 times less pencils than in the second one, and 5 times less than in the third one. How many pencils are there in each box?

2.17. There are 80 quarts of water in two barrels. If we pour 10 quarts of water from one barrel to the other one, then the first barrel will contain three times less water than the second. How much water initially was there in each of the barrels?

2.18. Wolf, Hedgehog, Sparrow, and Beaver were sharing an orange. Hedgehog got twice as many slices as Sparrow, Sparrow got five times less slices than Beaver, Beaver got 8 more slices than Sparrow. Into how many slices was the orange cut if Wolf got nothing?

2.19. Lisa and Bart Simpson ate up a barrel of jam and a basket of cookies, starting and ending simultaneously. At first Bart ate cookies while Lisa ate jam, and then (at some moment) they inverted roles. Lisa ate both the jam and the cookies three times faster than Bart. What part of the jam was eaten by Lisa if they ate an equal amount of cookies?

2.20. There are blue, green, and red pencils in a box, 20 in all. There are 6 times more blue pencils than green ones, and there are less red pencils than blue ones. How many red pencils are there in the box?

2.21. The Big Bad Wolf and the Three Little Pigs wrote a detective story entitled "Three Little Pigs–2"; then the same wolf, together with Red Riding Hood and her grandmother, wrote the cookbook "Red Riding Hood–2". The publishers gave all the royalties for the two books to one of the pigs. He took his share and gave the remaining 2100 gold coins to Wolf. The royalties for each book are divided equally between its authors. How many coins should Wolf take?

2.22. Three kinds of fish live in an aquarium: goldfish, silverfish, and redfish. If the cat eats up all the goldfish, there will be one fish less than two thirds of the initial total number of fishes. If the cat eats up all the redfish, there will be 4 fishes more than two thirds of their initial total number. Are there more or less goldfish than silverfish, and by how many?

Introducing the signs of arithmetical operations

Recall that if an expression contains no parentheses, then first we must perform the operations of multiplication and division from left to right, and after that, all the operations of addition and subtraction from left to right.

2.23. Place the signs "+" between parts of the expression 1234567 so that the sum should equal 100. Find two solutions.

2.24. Place the signs "+" between parts of the expression 987654321 so that the sum should equal 99. Find two solutions.

2.25. In the equation $2222 = 55555$ place signs of arithmetical operations (without using parentheses) so that it should become true.

2.26. What numbers can be obtained by placing one or several signs "+" between the digits of the number 77 777?

2.27. In the equation $1 * 2 * 4 * 8 * 16 * 32 * 64 = 27$ replace the stars by "+" or "−" so that it should become true.

Introducing parentheses

Recall that if an expression contains parentheses, the operations in the parameters must be performed first.

2.28. In the expression 1∗2∗3∗4∗5 replace the stars by signs of arithmetical operations (addition, subtraction, multiplication, and division) and introduce parentheses so that the resulting number should equal 100.
2.29. Add parentheses to the expression $7 \cdot 9 + 12 : 3 - 2$ so that the its value should be equal to a) 23; b) 75.
2.30. a) Obtain the number 37 by adding signs of arithmetical operations between the digits of the number 33333.

b) Obtain the number 37 by adding signs of arithmetical operations between the digits of the number 33333 differently and introducing parentheses.
2.31. Is it possible to place one of the signs "+", "−", "·", and "÷" between each pair of neighbouring digits of the number 22222222, and then add parentheses so as to obtain 100?
2.32. Add parentheses to the left-hand side of the equation

$$1 \div 2 \div 3 \div 4 \div 5 \div 6 \div 7 \div 8 \div 9 \div 10 = 7$$

so that it should become true.

Sums of consecutive numbers

2.33. Find the sum $1 + 2 + 3 + \ldots + 100$.
2.34. Find the sum $3 + 4 + 5 + \ldots + 49$.
2.35. Ten fishermen caught 19 fish of mass 100 g, 200 g, ..., 1900 g. How can one divide the fish between the fishermen so that each should get the same mass of fish?
2.36. The sum of five successive natural numbers equals 875. Find these numbers.
2.37. The sum of six different natural numbers equals 22. Find these numbers.
2.38. How many two-digit numbers whose first digit is greater than the second one are there?
2.39. Evaluate the expression

$$26 \cdot 25 - 25 \cdot 24 + 24 \cdot 23 - 23 \cdot 22 + 22 \cdot 21 - 21 \cdot 20 + 20 \cdot 19 - \\ - 19 \cdot 18 + 18 \cdot 17 - 17 \cdot 16 + 16 \cdot 15 - 15 \cdot 14.$$

2.40. The digits $1, 2, 3, \ldots, 9$ are arranged in a circle (not necessarily in this order). Any three consecutive numbers, if read in the clockwise direction, constitute a three-digit number. Find the sum of all the nine such numbers. Does it depend on the ordering of the digits $1, 2, 3, \ldots, 9$ in the circle?

Numerical rebuses

In each of the Problems 2.41–2.50 you must decipher numerical rebuses (i.e., replace different letters by different digits and identical letters by identical digits).

2.41.
$$\begin{array}{r} \mathrm{A\ B\ C} \\ +\ \mathrm{A\ C\ B} \\ \hline \mathrm{B\ C\ A} \end{array}.$$

2.42.
$$\begin{array}{r} \mathrm{A\ B\ C\ D} \\ +\ \mathrm{A\ B\ C\ D} \\ \hline \mathrm{B\ D\ C\ E\ C} \end{array}.$$

2.43.
$$\begin{array}{r} \mathrm{Z\ Y\ X\ W} \\ +\ \mathrm{Z\ Y\ X\ W} \\ \hline \mathrm{V\ W\ Z\ U\ Z} \end{array}.$$

2.44.
$$\begin{array}{r} \mathrm{H\ I\ J\ K\ L} \\ +\ \mathrm{H\ I\ J\ K\ L} \\ \hline \mathrm{M\ K\ M\ N\ I\ H} \end{array}.$$

2.45. $\mathrm{QRST} \times 4 = \mathrm{TQRS}$.

2.46. $\mathrm{UVWXYZ} \times 5 = \mathrm{ZYVWXY}$.

2.47.
$$\begin{array}{rrr} & & \mathrm{A} \\ + & \mathrm{A} & \mathrm{B} \\ \mathrm{A} & \mathrm{B} & \mathrm{C} \\ \hline \mathrm{B} & \mathrm{C} & \mathrm{B} \end{array}$$

2.48. $\mathrm{FGHIJ} \times 3 = \mathrm{ZJYFJ}$.

2.49. $\mathrm{ABCD} \times 18 = \mathrm{EFBGHA}$.

2.50. $\mathrm{ZYXYW} \times 5 = \mathrm{VUTZSR}$.

2.51. Peter wrote a sum on the blackboard, then replaced some digits by letters, with different digits replaced by different letters and identical digits replaced by identical letters. He obtained:

$$\mathrm{DEFGG} + 2011 = \mathrm{GHIEH}.$$

Prove that Peter made a mistake.

2.52. Find the smallest four-digit number ROOM for which the rebus

$$\mathrm{MY} + \mathrm{NEAT} = \mathrm{ROOM}$$

has a solution.

Choosing the right numbers

2.53. A four-digit number begins with 6. If one moves this digit to the end of the number, the resulting four-digit number will be 1152 less than the original one. Find the original number.

2.54. From four nonzero digits we constructed two four-digit numbers: the largest and the smallest one. The sum of these two numbers turned out to equal 11 990. What were the two constructed numbers?

2.55. Substitute digits for stars :

```
        * * 5
   ×      4 *
    ---------
        3 * *
 +    * 2 * *
    ---------
    1 * * * *
```

2.56. Substitute digits for stars :

```
        * 2 * 3
   ×        * *
    -----------
      * * * 8 7
 +  * * * * *
    -----------
    2 * 0 0 4 *
```

2.57. Substitute digits for stars :

```
              * * 8 * *
        ---------------
    * *) * * * * * * *
         * * *
         -----
             * *
             * *
             ---
               * * *
               * * *
               -----
                   0
```

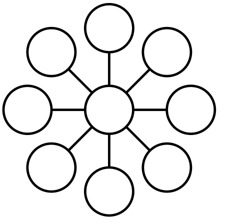
Verify that all the digits are uniquely determined.

2.58. Substitute digits for stars :

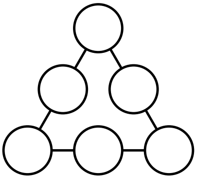
```
           * 3 *
         -------
    * 3) 3 * * *
         * 3
         ---
           * *
           * *
           ---
           * 3 *
           * * *
           -----
               0
```

Placing numbers in circles and cells

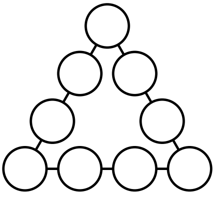
2.59. Place the digits from 1 to 9 in the circles (see the figure) so that the the sum of any three digits on one (straight) line should be 15.



2.60. Place the digits from 1 to 6 in the circles (see the figure) so that the sum of any three digits on each side of the triangle should be 10.



2.61. Place the digits from 1 to 9 at the vertices of the triangle, one digit at each vertex, and on its sides, two digits on each side, so that the the sum digits on each side should equal a) 17; b) 20.

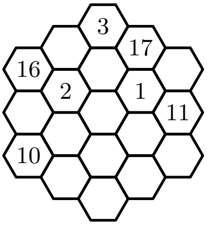


2.62. A sequence of digits and stars is written on the blackboard:

$$5,\ *,\ *,\ *,\ *,\ *,\ *,\ 8.$$

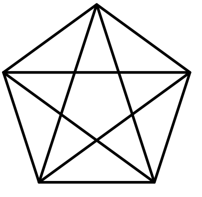
Replace the stars by digits so that the sum of any three successive digits should be equal to 20.

2.63. Fill in the cells using the missing numbers from 1 to 19 so that the sum of numbers along all the vertical and diagonal lines should be the same.



2.64. Place four 1's, three 2's, and three 3's in a circle so that the sum of any three successive digits should not be divisible by 3.

2.65. A pentagon is divided into 11 (one smaller pentagon and ten small triangles) parts by its diagonals. Place the numbers 1 to 11 in these parts so that the sum of numbers in each of the ten triangles formed by triplets of vertices of the pentagon should be the same.



Power of a number

The product of n factors, each of which is equal to the same number b (called the *base*), is called *b raised to the power of n* (briefly *b to the nth*) and denoted b^n; the positive integer n is the *exponent*. For $n = 0$, we set $b^n = 1$.

We obviously have:

$$a^n \cdot b^n = (ab)^n,$$

more generally:

$$a^n \cdot a^m = a^{n+m},$$

and also:

$$(a^n)^m = a^{nm}.$$

2.66. How many digits are there in the decimal expression of the number 10^{1000}?

2.67. How many digits are there in the decimal expression of the number 2^{20}?
2.68. Can four different powers of 2 contain the same number of digits?
2.69. Prove that for each natural number n the number of different n-digit powers of 2 is at least three.
2.70. Prove that for each natural number n the number of different n-digit powers of 2 is at most four.
2.71. Decipher the rebus: $\mathrm{A}^{\mathrm{B}} = \mathrm{BCDEA}$.
2.72. Decipher the rebus: $\mathrm{A}^{\mathrm{B}} = \mathrm{CAD}$.
2.73. Decipher the rebus: $\mathrm{AB}^{\mathrm{C}} = \mathrm{BDEF}$.

Magic squares

In a 3×3 *magic square* the numbers from 1 to 9 are placed so that the sum of numbers in any column is equal to the same number and the sum of numbers in any row is equal to that same number.

2.74. What is that sum equal to?
2.75. Construct a magic square with 5 at one of the corners.
2.76. Construct a magic square satisfying the following extra condition: the sum of the three numbers on any diagonal equals 15.
2.77. Is it possible to construct a magic square satisfying the extra condition from the previous problem such that the number in the central cell is not 5?
2.78. Place the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 in a 3×3 square so that the sum in each row, each column, and each diagonal should be the same.
2.79. Recover a magic square:

16	3	2	13
5	10	11	8
9	6		
4	15		

2.80. In a 4×4 square place the numbers 1 to 16 in increasing order. Interchange the numbers located at opposite corners, and then the numbers located at opposite corners of the inner 2×2 square. Write out the result and verify that it is a magic square.

Chapter 3

Word Problems

Motion

As a rule, we assume in this section that all the speeds are constant and the same throughout the problem. If this is not the case, it is said explicitly in the problem. For example, in Problem 3.9 the squirrel carrying the nut does not run at the same speed as without the nut.

3.1. The distance between the musketeers Athos and Aramis, galloping along the road, is 20 leagues. In one hour Athos covers 4 leagues, Aramis covers 5 leagues. What will the distance between them be one hour later?

3.2. Anna's trip to school and back by bus takes 30 minutes. The trip to school by bus and back by foot takes one hour and 30 minutes. How long will the trip to school and back by foot take?

3.3. Ken and his sister Mary went to visit a friend. Having covered one fourth of the route, it occurred to Ken that he has forgotten to take the gift prepared for his friend, so he turned back, while Mary went on. Mary arrived twenty minutes after leaving the house. How much later will Ken arrive?

3.4. There is 10 meters between a fox and a hare. When will the fox catch the hare if it runs 8 meters per second, while the hare runs 7 meters per second?

3.5. A messenger is sent from Moscow to Vologda. He covers 40 miles per day. Another messenger, who covers 45 miles per day, is sent after him. When will the second messenger catch up?

3.6. I walk to school from home in 30 minutes, my younger brother, in

40 minutes. In how many minutes will I catch up with my brother if he left the house 5 minutes before me?

3.7. A pedestrian started walking from point A to point B. At the same moment, a bicycle rider started riding from B to A. One hour later the pedestrian was exactly at the same distance from the cyclist as from point A. Fifteen minutes later they met and each continued on their way. How long will it take the pedestrian to get from A to B?

3.8. It takes Peter 20 minutes to get to school from home. One day, on the way to school, it occurred to him that he had left his pen at home. Now if he, nevertheless, continues on his way to school, he will get there 3 minutes early, and if he goes back for the pen, he will be 7 minutes late. What part of his route had he covered when he remembered the pen?

3.9. Without a nut (from the nest to the hazel tree) the squirrel runs at a speed 4 m/s, on the way back (carrying a nut) at 2 m/s. It takes him 54 seconds to go there and back. Find the distance between the nest and the tree.

3.10. A snail climbs 4 meters up a lamppost during a day, and slides down 2 meters at night. In how many days will it reach the top of the post, which is 8 meters high?

3.11. Two pedestrians walk towards each other at the speed of 5 kilometers per hour. When the distance between them is 10 kilometers, a fly sitting on the first pedestrian starts moving towards the second one at 14 km/h, reaches, and immediately begins flying back to the first pedestrian, then to the second one again, and so on. How many kilometers will the fly have covered when the pedestrians meet?

3.12. A car moves at 60 km/h. To what level should the driver increase the speed to gain one minute each kilometer?

3.13. Every day at noon a steamship leaves Le Havre for New York, and at the same time a ship of the same company leaves New York for Le Havre. Each of the ships is on its way for seven days, following the same route. How many ships from the same company will a ship going from Le Havre to New York meet on her way? (The ships that meet in the ports of Le Havre and New York don't count.)

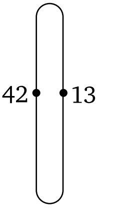
3.14. Johnny was walking along a straight road from one bus stop to another. Having covered one third of the distance, he looked back and saw a bus was nearing the stop that was behind him. In such situations Johnny is allowed to turn back. It is known that no matter to which of the two stops Johnny will run now, he will get there simultaneously with the bus. Find the speed of the bus if Johnny runs at 20 km/h.

3.15. Two frogs Croaky and Grunty race against each other: 20 meters straight ahead and back. Croaky covers 60 cm in one jump, Grunty, only 40 cm, but Grunty performs three jumps while Croaky does two. Who will win the race?

3.16. From the post office to the airport a car was sent to fetch mail. The airplane with the mail landed earlier than expected, so the arriving mail was sent to the post office in a passing truck. After 30 minutes on the road, the truck met the car, which picked up the mail and immediately turned back. The car returned to the post office 20 minutes earlier than expected. How many minutes earlier than on schedule did the plane land?

3.17. The classmates Ann, Boris, and Cyril live on the same stairwell. They go to school at constant, but different speeds, without looking back or waiting for each other. But if one of them catches up with another, the person catching up slows down to the speed of the person he has caught up with and they continue together. One day Ann was the first to leave, followed by Boris and then by Cyril. Two of them reached school at the same time. On the next day Cyril was the first to leave, followed by Boris and then Ann. Can it happen that the three get to school together?

3.18. The ski lift cabins at a mountain resort are numbered 1 to 99. Ian got on cabin № 42 at the lower terminal; at some moment he noticed that that he was facing cabin № 13 (see the figure) and 15 seconds later his cabin faced cabin № 12. How long will it take Ian to reach the upper terminal after he met cabin № 12?



3.19. Peter was eating in a roadside cafe when a bus rode past. Three spoonfuls later after the bus, a motorbike passed, and three more spoonfuls later, a car. Sam was eating at another cafe further down the road and watched the bus, bike, and car go past, but in a different order, first the bus, three spoonfuls later, the car, and three more spoonfuls later, the motorbike. It is known that Peter and Sam always eat at the same constant speed. Find the speed of the bus if the car's speed is 60 km/h and the speed of the motorbike, 30km/h.

3.20. Dmitry lives in a 9-floor apartment house. It takes him one minute to get to the ground floor from his flat by lift. Because Dmitry is short, he cannot reach the button for his floor. As the result, when returning home, he pushes the highest button that he can reach, and then walks up. The whole trip up takes 1 minute 10 seconds. The lift moves up and down at the same speed, and Dmitry moves up at half the speed of the lift. At what floor does Dmitry live?

Age

3.21. When Ivan was asked how old he is, he answered, "I am three times younger than my father, but three times older than my younger brother Serge." Then the younger brother appeared and said, "My father is forty years older than I." How old is Ivan?

3.22. Lisa is 8 years older than Nadine. Two years ago she was three times older than Nadine. How old is Lisa?

3.23. When his father was 31, the son was 8, but now the father's age is twice that of the son. How old is the son?

3.24. When his father was 27, the son was 3, but now the son's age is three times less than the father's. How old are they today?

3.25. The age of the granddaughter in months is equal to the grandmother's age in years. Their ages add up to 65 years. How old is the grandmother and the granddaughter?

3.26. Tina's 16th birthday was 19 months ago, while Mike will be 19 in 16 months. Who is older?

3.27. I am as many times younger than my grandfather as I am older than my sister. How old am I if my sister is not 7 yet, while my age with grandpa's add up to 84? All the quotients of ages are integer, all the ages, if measured in years, are also integer.

Joint work

3.28. Five cats caught 5 mice in 5 minutes. How many mice will ten cats catch in 10 minutes?

3.29. Johnny eats 600 grams of jam in 6 minutes, Davy eats jam twice as fast. How long will it take them to eat 600 grams of jam together?

3.30. Ten beavers figured out that they can build a dam in 8 days. Having worked for 2 days, it turned out that a flood is expected in 2 days so that they must finish the work before the flood occurs. How many more beavers must they call as help?

3.31. In one day two lumbermen can saw off three logs, or chop up six logs. How many logs should they saw off in order to be able to chop them all up in the same day?

3.32. One man can drink up one cask of a drink in 14 days alone, and together with his wife, in 10 days. In how many days can the wife drink up the same cask without her husband's help?

3.33. A dog and a cat simultaneously grabbed a sausage at opposite ends. If the dog bites off his part and runs away, then the cat will get 300 grams more than the dog, if the cat bites off his part and runs away, the dog will get 500 grams more than the cat. How much sausage will be left if both bite off their pieces and run away?

3.34. A construction firm hired two diggers to dig a tunnel. In an hour, one of them can dig twice as much as the other, but both get the same pay for an hour of work. What will be cheaper, joint work from both ends of the tunnel until they meet or digging of half the tunnel by each?

Purchases

3.35. One hundred rubles were paid for a book, plus half the price of the book. How much did the book cost?

3.36. In a pet shop large and small birds are sold. A large bird costs twice as much as a small one. One lady bought 5 large birds and 3 small ones, another lady bought 5 small birds and 3 large ones. The first lady paid 20 rubles more than the other one. How much do the small birds and the large birds cost?

3.37. One cup and one saucer together cost 25 rubles, four cups and three saucers cost 88 rubles. How much does one cup cost?

3.38. If a student buys 11 notebooks, he will have five rubles left, but he is 7 rubles short to buy 15 notebooks. How much money does he have?

3.39. In the market 10 doughnuts can be exchanged for 3 pieces of cheesecake, and one piece of cheesecake for 3 doughnuts and 5 rubles. How much does one piece of cheesecake cost?

3.40. A wandering merchant bought a batch of pens in a wholesale market; he sells them either for 5 rubles each or in threes, for 10 rubles a batch of three. The merchant makes an equal amount of money from the two types of customers. What is the wholesale price of one pen?

3.41. A customer in a store buys some goods for 20 rubles and gives the cashier a 100-ruble bill. The cashier sees that she hasn't the right amount of change. The customer went to another cash register, exchanged the 100-ruble bill and left the store with his purchase. The cashier at that cash register notices that the bill is phony. How much money did the store lose?

3.42. In circulation in the Catchfools city there are 1, 2, 3, ..., 19, and 20 soldo coins (and no others). Pinocchio had one coin, with it he bought an icecream cone and was given one coin as change. He then bought another icecream cone and got three coins of different denominations as change. He wanted to buy a third cone, but didn't have enough money. How much does an icecream cone cost?

Legs and heads

3.43. Together, several pups and ducklings have 44 legs and 17 heads. How many pups and how many ducklings are there?

3.44. Ten dogs and cats were fed 56 biscuits. Each cat got 5 biscuits, each dog, 6. How many dogs and how many cats were there?

3.45. Three friends, Peter, Tony, and Victor came up to a parking lot for motorbikes and cars. Peter, having nothing to do, counted all the vehicles — there were 45. Tony counted all the wheels — there were 115. Victor noticed that there was twice as many sidecar motorbikes as motorbikes without sidecars. How many cars were there at the parking lot?

3.46. In a room there are several four-legged chairs and three-legged stools. When one person sits on each of the chairs and one person sits on each of the stools, the total number of legs is 39. How many chairs and how many stools are in the room?

Sharing

3.47. Basil picked more mushrooms than his sister Helen. On the way home she started asking him, "Give me some mushrooms, so that I will have as many as you." Basil gave to his sister 10 mushrooms. How many mushrooms more than Helen did Basil pick?

3.48. Basil found 36 more mushrooms than his sister Helen. On the way home she started asking him, "Give me some mushrooms, so that I will have as many as you." How many mushrooms must Basil give to Helen?

3.49. A group of children went mushroom picking in the forest. If Ann gives half of her mushrooms to Victor, all the children will have the same number of mushrooms, while if she gives all her mushrooms to

Alex instead, he will have as many mushrooms as all the others together. How many children went mushroom picking?

3.50. Three workers sat down to eat. The first had 4 flapjacks, the second, 8, while the third contributed 15 rubles. The flapjacks were divided equally among the three. How should the first and second divide the money between themselves?

3.51. Three workers sat down to eat. The first had 6 flapjacks, the second, 9, while the third contributed 15 rubles. The flapjacks were divided equally among the three. How should the first and second divide the money between themselves?

What is bigger?

3.52. Six mackerels are heavier than ten herrings. What is heavier, two mackerels or three herrings?

3.53. Four black cows and three brown ones give as much milk in five days as three black cows and five brown ones give in four days. Which cows give more milk, the black ones or the brown ones?

3.54. 40 gold coins and 40 silver coins were placed into three empty chests, with both kinds of coins in each of the three chests. In the first chest, there were 7 more gold coins than silver ones, while the second chest contained 15 less silver coins than gold ones. Which type of coin is in the majority in the third chest and by how much?

3.55. In an animal exhibition every third cat is white and every sixth white animal is a cat. What is more, the number of cats or the number of white animals that are not cats?

3.56. Six mathematicians went fishing. Together they caught 100 fish, and no two of them caught the same quantity of fish. Afterwards they noticed that each one of them could distribute his whole catch among the five others so that these five would have the same number of fish. Prove that one of these mathematicians can go home with his entire catch so that each of the remaining five would be able to distribute his entire catch among the remaining four in such a way that these four would have the same quantity of fish.

Joint weight

3.57. Mary and her mother together weigh 10 kilograms more than Mary's father, Mary and her father together weight 50 kilograms more

than Mary's mother. How much does Mary weigh?
3.58. Ken and his mother together weigh 100 kg, Ken and his father, 120 kg, the parents together, 140 kg. How much do they weigh all three together and each separately?
3.59. A container full of milk weighs 33 pounds, and when it is half full, it weighs 17 pounds. How much does the empty container weigh?

Awards and penalties

3.60. Jeff went to a shooting gallery with his father. They agreed that Jeff makes 5 shots and then two more shots for each time he hits the mark. All together Jeff made 17 shots. How many times did he hit the mark?
3.61. Coming to the shooting gallery, Peter paid for 5 shots. For every successful shot he is allowed to make 5 more shots for free. Peter says that he made 50 shots and hit the mark 8 times, but his friend Basil says that this is impossible. Which one of the boys is right?
3.62. A participant in a quiz answered 20 questions, earning 86 points. Twelve points are given for each correct answer, ten points are deducted for each wrong answer. How many questions did this participant answer correctly?

Sawing and cutting

3.63. Two rabbits are sawing a log. They sawed through it 10 times. How many pieces of wood did they get?
3.64. To saw a log into two pieces, 5 minutes are needed. How long will it take to saw the same log into five pieces?
3.65. Several rabbits were sawing several logs. They sawed trough the logs 10 times, obtaining 16 pieces of wood. How many logs did they saw?
3.66. There are 35 logs, long and short. The long ones are sawed into 5 pieces, the short ones, into 4. To saw all the short ones, the number of cuts needed was the same as the number of cuts needed to saw the long ones. How many cuts were made?
3.67. Some of the 9 given sheets of paper were cut into three pieces. After that, the total number of sheets became 15. How many sheets were cut?
3.68. Snowwhite cut out a square cloth and put it in a chest. Then the First Gnome appeared, took out the square cloth, cut it into four

squares, and put them back into the chest. The Second Gnome took out one of the squares, cut it into four squares and put them back into the chest. Then came the Third Gnome, he also cut one of the squares into four and put it back. The next Gnomes did the same. When the Seventh Gnome left, how many squares of cloth were in the chest?

3.69. A rectangular three-by-four bar of chocolate is being broken into one-by-one square pieces. At each step only one break is performed (pieces of chocolate cannot be put one above the other.)

a) What is the least numbers of breaks needed to do that ?

b) The greatest number?

3.70. We have several 4 metre logs and 5 metre ones. How many logs of these types must we cut in order to obtain 42 one-metre-long logs, making the least number of cuts?

3.71. On the skin of a sausage thin transverse circles are drawn. If one cuts along the red circles, 5 pieces are obtained, along the blue circles, 7, along the green circles, 11. How many pieces will be obtained if one cuts along all the colored circles?

3.72. In Tania's flat there are 8 sockets, 21 tees (each tee has three sockets and one plug) and a boundless amount of irons. What is the largest number of clothes irons that Tania can plug in simultaneously?

Exchange

3.73. In a fairytale kingdom there are two exchange offices. In the first, you can exchange one ruble for 3000 soldos, but they take a 7000 soldo commission for the exchange, while the second office gives only 2950 soldo for a ruble and charges nothing for the exchange. A tourist figured out that he can make the exchange equally well in the two offices. How many rubles did he intend to change?

3.74. A rich Mole acquired 8 sacks of grain last fall. During each of the three winter months he needs 3 sacks of grain or 1 sack of grain and 3 sacks of millet. Mole can exchange 1 sack of grain for 2 sacks of millet with other moles. But his den can hold no more than 12 sacks, and in the winter Mole cannot leave his den and so can't perform any exchanges. Help Mole to make adequate preparations for the winter.

3.75. At an exchange office two operations are performed:

1) you give 2 euro and get 3 dollars and a chocolate;

2) you give 5 dollars and get 3 euro and a chocolate.

When Pinocchio came to the exchange office, he only had dollars, but when he left, he had less dollars, he had acquired no euros, and had 50 chocolate candies. How many dollars did he lose?

Thirteen more problems

3.76. Two TV channels simultaneously began showing the same film. The first channel divided the film into several 20-minute parts and introduced two-minute ads between them. The second channel divided the film into 10-minute parts with one-minute advertisements in between. On which channel will the film end first?

3.77. Pete and Basil took part in a bicycle race. All the participants started at the same time and finished at different times. Pete finished right after Basil and turned out to be in 10th place. How many cyclists participated if Basil was 15th from the end?

3.78. Exactly three runners participated in a separate cross country race. Greg was the first among them to start, then, Alex, and after him, Ellen. After the finish, it turned out that that during the race Greg overtook one of the two others 10 times, Ellen overtook one of the two others 6 times, Alex, 4, and all three were never at the same place at the same time. In what order did the three finish if it is known that they finished at different times?

3.79. An alarm clock is 9 minutes fast a day (24 hours). Going to bed at exactly 10pm, the clock was set to the right time. For what time should the alarm be set so that it will ring at exactly 6am?

3.80. The numbers of all the pages of a book (every page, from the first to the last, has a number) contain 1392 digits all told. How many pages are there in the book?

3.81. Peggy was sitting on a bench in the park not far from the bell tower, reading an interesting book. The bell rings every hour on the hour the appropriate number of times, and also rings once on each half hour. While Peggy was sitting on the bench, she heard 5 distinct series of bell strikes (one isolated strike is also a series), amounting to a total of 11 strikes. After the last bell strike, she went home. What was the time when she left?

3.82. Nadine baked raspberry, blueberry, and strawberry muffins. The number of raspberry muffins was half the total number of muffins; the number of blueberry muffins was 14 less than that of raspberry muffins, while the number of strawberry muffins was half the number of raspberry and blueberry muffins taken together. How many muffins of each type did Nadine bake?

3.83. In a clearing in the forest there were 35 white and yellow dandelions. After 8 of the white ones lost their seeds to the wind and two yellow ones turned white, there were still twice as many yellow ones than white ones. How many yellow and how many white dandelions were there in the clearing at the beginning?

3.84. Dandelions start to bloom in the morning, becoming yellow blossoms and staying yellow for two days. On the third morning they become white and lose their seeds in the evening. Yesterday in the early afternoon there were 20 yellow and 14 white ones in the clearing, and today there are 15 yellow ones and 11 white ones.

a) How many yellow dandelions were there in the clearing the day before yesterday?

b) How many white ones will there be tomorrow?

3.85. Winnie-the-Pooh has five friends, each one of them has pots of honey at home: Tigger has one, Piglet has two, Owl, three, Eyore, four, and Rabbit, five. Winnie-the-Pooh visits his five friends in turn, eats one pot of honey and takes the remaining ones with him. He reached the home of his last friend carrying 10 pots of honey. Which one of his friends could he have visited first?

3.86. Thirty five crows flew into the clearing. Unexpectedly, they all flew up, forming two flocks. One flock installed itself on the branches of a birch, the other on an alder. After a while, 5 crows flew from the birch to the alder, the same number flew away from the birch, and then the number of crows remaining on the birch was twice that of the crows remaining on the alder. How many crows were there in each of the two initial flocks?

3.87. The chipmunks Chip and Dale must save the same number of nuts for the winter. After Chip had hoarded 120 nuts, and Dale, 147, Chip still had to save four times more nuts than Dale. How many nuts does each have to save?

3.88. Ken planted a certain number of bulbs along a garden path. Then Tina planted one more bulb between each pair of bulbs planted by Ken. Then came Inna and planted a new bulb between all the bulbs planted before her. Then Doug came and did the same. All the bulbs grew, and 113 tulips bloomed. How many bulbs did Ken plant?

Chapter 4

Divisibility of Natural Numbers

Even and odd

4.1. Each one of seven gnomes has one red balloon and one yellow balloon. Can they exchange balloons so that each all of them has two balloons of the same color?

4.2. Is it possible to place all the 28 domino tiles in a line such that the values of the adjacent tile ends should match and that the values at the ends of the line should be 5 and 6?

4.3. King Arthur wrote the number 20 on a piece of parchment. Each one of the 33 knights of the Round Table in turn either added 1 or subtracted 1 from the written number. Can the number 10 be obtained as the result?

4.4. a) The sum of two natural numbers is even. Prove that their difference is also even.

b) The sum of two natural numbers is odd. Prove that their difference is also odd.

4.5. Prove that the sum of an odd number of odd numbers is odd.

4.6. The A-student Paul bought a 192-page notebook and numbered its pages from 1 to 192. Ken, the worst student in the class, tore out 25 sheets of paper from Paul's notebook and added the 50 numbers written on them, obtaining 2002. Didn't Ken make a mistake?

4.7. The 5th form pupil Kathy and some of her classmates form a circle holding hands. In that circle each pupil either holds hands with two boys or with two girls. If there are 5 boys in the circle, how many girls are there in it?

4.8. In a competition for middle school pupils there were contests in math, physics, chemistry, biology, and ballroom dancing. When the competition was over, it turned out that an odd number of pupils participated in each contest and each pupil took part in an odd number of contests. What is the parity of the number of pupils that participated in the competition?

4.9. The inhabitants of Mars can have an arbitrary number of hands. Once upon a time all the Martians held hands so that no free hands remained. Can the number of Martians who have an odd number of hands be odd?

4.10. Six numbers are written on the blackboard: 1, 2, 3, 4, 5, 6. In one step, we are allowed to simultaneously add 1 to two of these numbers. Can we make all the numbers equal after several steps?

4.11. The numbers m and n are integers. Prove that the number $mn(m+n)$ is even.

4.12. Is it possible to find four integers whose sum and product are odd?

4.13. Seven natural numbers are written in a circle. Prove that there are two neighbouring numbers whose sum is even.

4.14. There are 20 shawls on several hangers. Seventeen girls come up to the hangers in turn, each one either takes or hangs up exactly one shawl. Can 10 shawls remain on the hangers after the girls leave?

4.15. In the relation $1*2*3*4*5*6*7*8*9*10=0$ can one replace the stars by plus or minus signs so that the relation becomes true?

4.16. Five numbers are given such that the sum of any three of them is even. Prove that all these numbers are even.

4.17. From a complete 28 tile set of dominoes all tiles with sixes were thrown out. Can we construct a single line of dominoes such that the values of the adjacent tile ends should match, using all the remaining tiles?

4.18. Nicky took a book, counted how many digits are needed to number all the pages starting from the first, and found out they were 100. Didn't he make a mistake?

4.19. In some country there are two chambers in Parliament with the same number of members. All the MP's took part in the vote on an important question, and none abstained. When the chairman said that the resolution was accepted with an advantage of 23 votes, the leader of the opposition declared that the vote was falsified. How did he come to that conclusion?

4.20. The numbers from 1 to 20 are written out in a row. Two players in turn place pluses or minuses between them. When all the intervals between successive numbers are filled with pluses or minuses, the resulting number is calculated. If this number is even, the first player wins, if it is odd, the second one wins. Which player has a winning strategy?

4.21. N knights in armour from two warring kingdoms sit at a round table. The number of neighbouring allied pairs is equal to the number of neighbouring enemy pairs. Prove that N is divisible by 4.

4.22. Twenty five young boys and twenty five young girls sit around a round table. Prove that both neighbours of one of the children are boys.

4.23. What is the largest quantity of numbers that can be written in a sequence so that the sum of any 17 successive numbers should be even while the sum of any 18 successive numbers should be odd?

Divisibility tests

4.24. How many zeros stand at the end of the product of all the numbers from 10 to 25?

4.25. Is the number $11 \cdot 21 \cdot 31 \cdot 41 \cdot 51 - 1$ divisible by 10?

4.26. A cowboy named Bill walked into the bar and asked the barman for a three-dollar bottle of whiskey and six boxes of waterproof matches, whose price he did not know. When the barman requested 11 dollars 80 cents for the purchase, Bill drew his six-shooter. Then the barman recalculated the price of the purchase and corrected the error. How did Bill figure out that the barman tried to cheat him?

4.27. Nine identical sparrows eat up less than 1001 grains a day, while ten such sparrows, more than 1100 a day. How many grains a day does one sparrow eat?

4.28. Prove that a natural number expressed in two digits or more is divisible by 4 if and only if the number represented by its last two digits is divisible by 4.

4.29. In the expression $59{*}4{*}$ replace the stars by digits so as to make the obtained number divisible by 36. Find all the solutions.

4.30. To the right and to the left of the number 10, attach the same digit so that the obtained four-digit number should be divisible by 12.

4.31. Natasha had to solve the following homework problem: express the given number of hours in minutes. She neatly wrote out the correct answer in her notebook and immediately closed it. In the class, she saw that the answer was smudged: two digits became blots: 234∗2∗0. What number had been written in the notebook?

4.32. Replace the stars in the expression 72∗3∗ by digits so that the obtained number should be divisible by 45.

4.33. The number 82∗∗ is divisible by 90. Find the quotient.

4.34. Prove that all the numbers:

a) 12 345 678; b) 987 654; c) 1 234 560; d) 98 765 445

are not perfect squares.

4.35. Recover the missing digits:

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 = 399{*}68{*}{*}.$$

4.36. Is there a power of 2 whose expression contains an equal number of zeros, ones, ..., nines?

Division without remainder

4.37. Victor had 3 times as many peanuts as Paul. He gave some of them to Paul (no more than 5) and divided the remaining peanuts equally between three squirrels. How many nuts did Victor give to Paul?

4.38. A family of 24 mice lives in a burrow. Each night exactly 4 of them go to a storehouse to get some cheese. Can it happen that at some time each mouse has visited the storage house with each of the other mice exactly once?

4.39. Can the following relation

$$\mathrm{L} \times \mathrm{O} \times \mathrm{G} = \mathrm{A} \times \mathrm{R} \times \mathrm{I} \times \mathrm{T} \times \mathrm{H} \times \mathrm{M},$$

be true if the letters are appropriately replaced by digits from 1 to 9? (Different letters are replaced by different digits.)

4.40. The number 444 is divisible by 3. Can any other number all of whose digits are fours be divisible by a number whose decimal representation consists only of threes?

4.41. Can the sum of three different natural numbers be divisible by each of the summands?

4.42. There were 15 muffins on a plate. Tom took three times as many muffins as Peter, and Johnny, three times less than Peter. How many muffins remained on the plate?

4.43. The sum of three smallest divisors of the number A equals 8. How many zeros stand at the end of A?

4.44. Rita, Louise, and Valerie were solving problems. To motivate their work, they bought some chocolates and agreed that for each solved problem the one who solved it first would get 4 chocolates, the second one, 2, and the third, 1. All the girls solved all the problems and each one earned 20 chocolates, and there were no simultaneously obtained solutions. Didn't they make a mistake in distributing the sweets?

4.45. A heap of 637 seashells lies on the table. One shell is removed and the remaining ones are divided into two heaps (not necessarily with the same quantity of shells). Then one shell is removed form one of the resulting heaps containing two shells or more, and the latter is divided into two and so on. Can it happen that at some moment each the heaps consists of exactly three shells?

4.46. Prove that the number $16^{11} - 2^{39}$ is divisible by 31.

4.47. Prove that the number $333^{777} + 777^{333}$ is divisible by 37.

4.48. Prove that the number $10^{23} + 10^{19} - 182$ is divisible by 18.

4.49. Prove that, for any 18 consecutive numbers, one of them is divisible by 18.

4.50. Prove that at least one of 18 consecutive three-digit numbers is divisible by the sum of its digits.

4.51. Two identical numbers were written on the blackboard. Steve attached the digits 100, from the left, to the first number, and he attached the digit 1 from the right to the second one. As the result, the first number became 37 times greater than the second one. What numbers were initially written on the board?

4.52. Find the smallest number such that sum of its digits is divisible by 17 and the sum of digits of the next number is also divisible by 17.

4.53. Find a five-digit number divisible by 101 all of whose digits differ.

4.54. Find a number that has exactly four divisors (including 1 and the number itself).

4.55. Find the smallest even three-digit number divisible by 37.

4.56. The digits of a three-digit number are written in inverse order and the smaller number is subtracted from the larger one. Prove that the difference is divisible by 9 and by 11.

4.57. Strolling on the road one morning, Fred and Eddie found a stack of 11-dollar bills each. In the tearoom down the road, Fred had 3 cups of tea, 4 doughnuts, and 5 muffins. Eddie drank 9 cups of tea and ate 1 doughnut and 4 muffins. Fred could pay for his meal using 11-dollar bills without change. Prove that so could Eddie.

4.58. In the 100-digit number 12345678901234…7890 all the digits located at the odd positions were deleted. In the obtained 50-digit number the same operation was performed, and so on. This process was continued while there was something to delete. What was the last deleted digit?

4.59. Yesterday Nick bought some ballpoint pens, black ones for 9 rubles each and blue ones, for 4 rubles. Returning to the store today, he noticed that the price tags for the pens had changed: the black ones now cost 4 rubles each, the blue ones, 9 rubles. He was annoyed. He declared, "Had I made that purchase today, I would have saved 49 rubles!" Is he right?

4.60. Give an example of a number N such that of the four assertions "N is divisible by 2", "N is divisible by 4", "N is divisible by 12", "N is divisible by 24", three are correct and one isn't.

4.61. Of the assertions "N is divisible by 2", "N is divisible by 4", "N is divisible by 12", "N is divisible by 24" three are correct and one isn't. Which one?

4.62. Concerning seven natural numbers it is known that the sum of any six of them is divisible by 5. Prove that each of them is divisible by 5.

4.63. In a school there are 450 pupils and 225 desks (each desk is occupied by two pupils). Exactly one half of the girls share a desk with a boy. Is it possible to relocate the pupils so that exactly one half of the boys share a desk with a girl?

4.64. Prove that the square of a number that ends with the digit 5, ends with 25.

4.65. Find the largest four-digit number that is divisible by 7 and is expressed by four different digits.

4.66. Nicholas with his son and Peter with his son went fishing. Nicholas caught as many fish as his son, and Peter caught three times as many as his son. A total of 25 fish were caught. What is the name of Peter's son?

4.67. Are the numbers 1001 and 100 001 divisible by 11?

4.68. Prove that if to a three-digit number we append the three-digit number with the same digits but in inverse order, then the obtained number will be divisible by 11.

4.69. Using each of the digits 0 to 9 exactly once write out 5 nonzero numbers each one of which is divisible by the previous one.

4.70. In Moscow, bus tickets used to have six-digit numbers (possibly with leading zeroes). Let us call a bus ticket "lucky" if the sum of the six digits of its number is divisible by 7. Can two successive tickets be lucky?

4.71. Can we place in a circle the numbers:

a) from 1 to 7, b) from 1 to 9

so that each of them should be divisible by the difference of its two neighbours?

4.72. The teacher wrote a two-digit number on the blackboard and asked Daniel successively to answer the questions: is the number divisible by 2? by 3? by 4? ... by 9? Daniel answered all the eight questions correctly, and there was an equal number of "YES" answers and "NO" answers.

a) Can you correctly answer at least one of the teacher's questions without knowing the number?

b) At least two?

4.73. Johnny has five cards with the numbers 1,2, 3, 4, and 5 written on them. Help him to assemble two numbers (one with 3 digits, the other with 2) using the cards as digits so that the first number is divisible by the second.

4.74. Ken claims that one can find out whether the sum of all four-digit numbers not containing the digits 0 or 9 is divisible by 101 without calculating that sum. Is he right?

4.75. In a storeroom there were several whole wheels of cheese. During the night, a group of rats ate up 10 of them (all the rats ate the same amount). Next night, only 7 rats returned (there were rats who didn't) and ate up the rest of the cheese. This time each of the 7 rats ate up only half of what it had eaten the night before. How much cheese was there initially on the storeroom?

4.76. The consecutive natural numbers 2 and 3 are divisible by the consecutive odd numbers 1 and 3, the numbers 8, 9, and 10 are divisible by 1, 3 and 5 respectively. Are there 11 consecutive natural numbers which are divisible by 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, and 21, respectively?

Division with remainder

4.77. Find the largest integer which gives the incomplete quotient 5 under division by 7 with remainder.

4.78. Find all the natural numbers such that under division by 7 their incomplete quotient equals the remainder.

4.79. Two hikers started their trip at noon on Monday and returned in 100 hours. On what day and at what hour did they return?

4.80. Among the natural numbers that under division by 3 give the remainder 1, find the first three numbers that are divisible by 5.

4.81. Brownies live in a seven-floor apartment house. The elevator runs back and forth between the first and last floor, stopping at every floor. At each floor, beginning with the first, a brownie gets in the elevator and nobody gets out. On what floor will the thousandth brownie enter?

4.82. Is it true that among any four consecutive natural numbers, at least one is divisible: a) by 2; b) by 3; c) by 4; d) by 5?

4.83. Prove that, given a natural number, one can append a digit to it so that it becomes divisible by 9.

4.84. Is it true that any whole number of rubles greater than 7 can be paid by 5 and 3 ruble bills without change?

4.85. Under the division of a certain number m by 13 and 15, the quotients are the same, but the first division yielded the remainder 8, while the second one, no remainder. Find m.

4.86. One has 6 sacks weighing 13, 15, 18, 26, 31, and 36 kg. Each sack contains either salt, or flour, or sand. There is twice as much sand as salt in the sacks, and there is flour in only one sack. What are the contents of the sacks?

4.87. Can a natural number, under division by 9, yield the remainder 5, and under division by 6, the remainder 4?

4.88. Prove that the square of an odd number has the remainder 1 under division by 4.

4.89. Prove that the square of an odd number has the remainder 1 under division by 8.

4.90. Find the remainder under the division of the product of two consecutive numbers by the number that comes next after the greater of them.

4.91. Prove that the square of an odd natural number cannot end in two odd digits.

4.92. Prove that the product of six consecutive natural numbers is divisible by 9.

4.93. Find all the two-digit numbers that are divisible by each digit in their expression.

4.94. Last digits of two numbers are different. Can the cubes of these numbers have equal last digits?

4.95. Replacing one digit of the cube of a number by a star, one obtained 19∗83. What was the replaced digit?
4.96. Nineteen rose bushes are planted in a circle. Is it always possible to find two neighbouring bushes whose total number of blossoms is divisible by 3?
4.97. Prove that the square of a number cannot have the remainder 2 under division by 3.
4.98. Prove that the square of a number cannot have remainder 2 or 3 under division by 5.
4.99. Can the number $n^2 + n + 1$ be divisible by 5 for some natural n?
4.100. a) Prove that among any six distinct integers there are two whose difference is divisible by 5.

b) Will this assertion remain true if the word "difference" is replaced by "sum"?
4.101. One thousand chocolates are laid out in a row. Basil ate the 10th sweet from the left, and then he ate each 7th sweet further to the right. Then Peter ate the seventh (from the left) of the remaining sweets and after that he ate each 9th remaining sweet, also moving to the right. How many chocolates will remain after that?
4.102. Let us count the fingers on the right hand: first the pinkie, second, the ring finger, third, the middle finger, fourth, the index finger, fifth, the thumb, sixth, the index finger again, seventh, the middle finger, eighth, the ring finger, ninth, the pinkie, and so on. What finger will have the number 2020 in this enumeration?
4.103. The number a has the remainder 68 under division by 1967 and the remainder 69 under division by 1968. Find the remainder in the division of a by 14.
4.104. All the numbers divisible by 3 and not divisible by 7 have been removed from the list of numbers 1 to 333, as well as the all the numbers divisible by 7 and not divisible by 3. How many numbers remain?
4.105. The sum of digits of a three-digit number divisible by 7, is divisible by 7. Prove that its last two digits have identical remainders under division by 7 .
4.106. The sum of digits of a three-digit number divisible by 7, equals 7. Prove that its last two digits are identical.
4.107. The sum of digits of a three-digit number all of whose digits differ is divisible by 7, and so is the number itself. Find all such numbers.
4.108. Find the digits B, A, and O so that the number BAOBAB should be divisible by 101.

Prime numbers

4.109. Can the sum of three consecutive natural numbers be a prime number?

4.110. The number p is prime, $p > 3$.

a) Is it true that the numbers $p + 1$ and $p - 1$ are even?

b) Is it true that at least one of them is divisible by 3?

4.111. The number p is prime, $p > 3$.

a) Is it true that at least one the numbers $p+1$ and $p-1$ is divisible by 4?

b) Must at least one of them be divisible by 5?

4.112. Prime numbers have only two divisors, 1 and the number itself. What numbers have exactly three different divisors?

4.113. The numbers $2, 3, 4, \ldots, 29, 30$ are written on the blackboard. For one ruble we can underline one of the numbers. If a number is already underlined, we can underline all its divisors and multiples free of charge. What is the least sum one should pay to underline all the numbers on the board?

4.114. The number p is prime, $p > 3$. Prove that it has the form $6n+1$ or $6n - 1$, where n is a natural number.

4.115. Find all the natural numbers p for which p and $5p+1$ are prime.

4.116. Find all the pairs of prime numbers whose sum and difference are also prime.

4.117. a) Indicate a prime number that can be represented both as the sum of two prime numbers and as the difference of two prime numbers.

b) In this setting, what prime number must occur both in the expression for the sum and in the expression for the difference?

c) How many prime numbers satisfy the hypothesis of item a)?

4.118. A four-digit number all of whose digits are identical has only two prime divisors. Find that number and its prime divisors.

4.119. John Smith rented a flat in a building under construction which has five identical entrances. Initially the entrances were numbered from left to right and Smith's flat was number 636. When the builders changed the numeration of the entrances from 12345 to 54321, Smith's flat acquired the number 242. How many flats are there in the house? (The numeration of the flats accessible from the entrances did not change.)

4.120. A group of hikers is sharing cookies. It they divide two identical boxes of cookies equally, then one extra cookie will remain. If they divide three such boxes equally, then there will be 13 extra cookies. How many hikers are there in the group?

4.121. The father tells his son:

— Today is the birthday of both of us, and you are now half my age.

— Yes, and this is the eighth time in my life when your age is divisible by mine.

How old is the son, if the father is not older that 75?

4.122. In a magic square the sum of numbers in each row, in each column, and each diagonal is the same. Is it possible to construct a 3×3 magic square using the first nine primes?

4.123. The product of two natural numbers, both of which are not divisible by 10, equals 1000. Find their sum.

4.124. Find all the prime numbers p such that $p^2 + 2$ is also prime.

4.125. The numbers p and $p^2 + 2$ are prime. Prove that $p^3 + 2$ is also prime.

4.126. Find all the prime numbers p such that $2p^2 + 1$ is also prime.

4.127. Find all the prime numbers p such that $p^2 + 4$ and $p^2 + 6$ are also prime.

4.128. Prove that if $p > 3$ is prime then $p^2 - 1$ is divisible by 24.

4.129. Prove that if p and q are prime and $p > q > 3$, then $p^2 - q^2$ is divisible by 24.

4.130. Is it true that for all natural numbers n the number $n^2 + n + 41$ is prime?

4.131. Find all the prime numbers that cannot be represented as the sum of two composite numbers.

4.132. There are six children in a family. Five of them are, respectively, 2, 6, 8, 12, and 14 years older than the youngest one, and the age of each child is prime. How old is the youngest child?

4.133. A group of several friends were exchanging letters, each letter being sent to all of them except the sender. Each of the friends wrote the same number of letters, and as the result a total of 440 letters were received. How many people were there in the group of friends?

4.134. The numbers p and q are prime and distinct. How many divisors have the numbers: a) pq; b) p^2q; c) p^2q^2; d) p^mq^n?

4.135. Johnny wrote down a four-digit number. Lena added 1 to the first digit, 2, to the second one, 3, to the third, 4, to the fourth, and then multiplied the obtained sums. The product was equal to 234. What number could have been written by Johnny?

4.136. Indicate 10 consecutive natural numbers among which

a) there is no prime number;

b) there are precisely two primes; c) precisely three primes;

d) precisely four primes.

e) Check that among the numbers $2325, \ldots, 2334$ there is only one prime.

f) In general, how many primes among 10 consecutive numbers can there be?

4.137. Prove that there are infinitely many primes.

4.138. One of three prime numbers is equal to the difference of the cubes of the other two. Find this number.

Prime factorisation

4.139. Represent the number 203 as the sum of several natural numbers so that the product of these numbers also equals 203.

4.140. A group of classmates decided to go on vacation to the Russian town of Uglich. Every month each pupil had to contribute the same whole number of rubles, the same sum for each, and during 5 months they collected 49 685 rubles. How many pupils were there and how much did each contribute?

4.141. Indicate five natural numbers whose sum is 20 and whose product is 420.

4.142. Solve the rebus: $\mathrm{BAO} \times \mathrm{BA} \times \mathrm{B} = 2002$.

4.143. Solve the rebus: $\mathrm{AX} \times \mathrm{YX} = 2001$.

4.144. In all the entrances of an apartment house there is the same number of floors, and on each floor there is the same number of flats. In the house there are more floors than flats on each floor, the number of flats is more than the number of entrances, and there is more than one entrance. How many floors does the house have if it has a total of 105 flats?

4.145. Several teams consisting of the same number of night watchmen slept the same number of nights. Each watchman slept more nights than there are watchmen in each team, but less than the number of teams. How many watchmen are there on the teams if all together they slept a total of 1001 nights?

4.146. Find the smallest natural number which becomes a square when multiplied by 2 and becomes a cube when multiplied by 3.

4.147. At the end of the trimester Ivan wrote down in a row his current grades in music and put multiplication signs between some of them.

The product of the obtained numbers turned out to be 2007. What was his final grade in music? (The possible grades are 5 (excellent), 4 (good), 3 (satisfactory), 2 (unsatisfactory).) The final grade is obtained by rounding the average grade to the nearest integer.

4.148. A number was multiplied by the sum of its digits, obtaining 2008. Find it.

4.149. a) Scrooge keeps his gold coins in six chests. One day, counting his coins, he noticed that if he opens any two chests, then he can redistribute the coins into these two chests so that they will contain an equal number of coins. He also noticed that if he opens any 3, 4, or 5 chests, then he can redistribute the coins in them so that they will contain an equal number of coins. Then he heard a knock on the door, and the old miser never found out if it is possible (or not) to redistribute all the coins equally in the 6 chests. Can we answer that question without looking into the chests?

b) Suppose there are 77 chests and Scrooge can redistribute the coins equally in any 2, 3 ,4, 5, 6, ..., 76 chests. Is it possible (or not) to redistribute all the coins equally in the 77 chests?

GCD and LCM

The greatest common divisor of the numbers m and n is denoted by $\mathrm{GCD}(m,n)$, while the least common multiple of m and n is denoted by $\mathrm{LCM}(m,n)$.

4.150. The chocolates "Sweet Mathematics" are sold in boxes of 12. The candy bars "Geometry with Nuts", in boxes of 15. What least number of boxes of the chocolates of both types should one buy so as to obtain the same number of chocolates of the two types?

4.151. Nick, Serge, and Ian regularly went to the cinema. Nick would go there every fourth day, Serge, every fifth, Ian, every sixth. On March 31st all three went to the cinema. On what date will all three meet in the cinema next time?

4.152. Indicate two consecutive numbers the first of which has 8 as its sum of digits while the second one is divisible by 8.

4.153. There are less than 50 pupils in a class. In a test, one seventh of the pupils were given A's, one third, B's, half, C's. The rest of the tests were marked "unsatisfactory". How many unsatisfactory papers were there?

4.154. Find the GCD of 111 111 and 111 111 111.

4.155. Find the least natural number that yields the remainder 1 under division by 2, 2, under division by 3, ..., 7, under division by 8.

4.156. If we subtract 6 from a certain three-digit number, it will be divisible by 7, if we subtract 7, it will be divisible by 8, while if we subtract 8, it will be divisible by 9. Find this number.

4.157. Find the LCM and the GCD of the numbers 121 212 121 212 and 121 212.

4.158. The difference of two odd numbers is 8. Prove that these numbers are coprime.

4.159. Prove that for a natural number n the number $n(n+1)$ cannot be a perfect square.

4.160. Find the GCD of all the five-digit natural numbers expressed via the digits 1, 2, 3, 4, 5 without repetitions.

4.161. Find the GCD of all the nine-digit numbers expressed via the digits $1, 2, 3, \ldots, 9$ without repetitions.

4.162. The numbers 100 and 90 were divided by the same number. In the first case the remainder was 4, in the second case it was 18. What was the divisor?

4.163. Among the first 2000 numbers find three distinct numbers whose GCD is the largest possible.

4.164. Find all the pairs of natural numbers whose sum is 288 and GCD is 36.

4.165. The LCM of two numbers not divisible by each other is 630, their GCD is 18. Find them.

4.166. Do there exist six consecutive natural numbers such that the LCM of the first three is greater than the LCM of the next three?

4.167. Prove that if two numbers have the same remainders under division by the numbers a, b, c, ..., then these numbers give the same remainders under division by the LCM of a, b, c, ...

4.168. Scrooge keeps his gold coins in eight chests. One day, counting his coins, he noticed that if he opens any two chests, it is possible to redistribute the coins equally into the two chests. He also noticed that if he opens any 3, 4, 5, 6, or 7 chests, it is possible to redistribute the coins equally into the 3, 4, 5, 6, 7 chests. At that point he heard a knock on the door, and the old miser did not find out if it is possible to redistribute all the coins equally into 8 chests. Can you answer that question for him without looking into the chests?

Chapter 5

Fractions

The notation $\frac{3}{7}$ stands for an *ordinary fraction* or briefly, *fraction*. The number above the bar is the *numerator*, the one below the bar is the *denominator*.

A fraction is called *proper* if its numerator is smaller than its denominator, and *improper* if its numerator is greater than its denominator.

Improper fractions are often written as *mixed fractions*. For example, $\frac{16}{5} = 3 + \frac{1}{5}$ can be written as $3\frac{1}{5}$.

A fraction is called *reducible* if its numerator and denominator have a common multiple (other than 1). A fraction is called *irreducible* if its numerator and denominator have no common divisors other than 1.

Properties of fractions

5.1. When a bicycle rider covered $\frac{2}{3}$ of his route, he got a flat. The remainder of his trip on foot took twice as long as on the bicycle. How many times faster did the cyclist ride than walk?

5.2. Three sprinters, Alex, Serge, and Tony, participate in a 100-metre race. When Alex reached the finish line, Serge was 10 metres behind. When Serge crossed the finish line, Tony was 10 metres behind. What was the distance between Tony and Alex when Alex finished? (The three runners move at constant, but different speeds.)

5.3. The volume of each of three pails is an integer number of litres. If we pour all the water from the full first pail into the second one, the latter will be $\frac{2}{3}$ full, and if we pour it into the third one, it will be $\frac{3}{4}$ full.

Now if we pour the contents of the three full pails into a 30-litre barrel, then the barrel will not be full. How many more litres can be poured into it?

5.4. Ten books cost more than 11 dollars, 9 books cost less than 10 dollars. How much does one book cost?

5.5. In an octopus family there was an equal number of white, blue, and green baby octopuses. When several blue octopuses turned green, their father decided to count his children. According to his count, the total number of blue and white children is 10, and the total number of white and green children is 18. How many baby octopuses are there in the family?

5.6. The ship hold of a steamer started to leak. A pump was turned on immediately, but this was not sufficient, and in 10 minutes the level of water in the hold rose by 20 cm. Then a second pump of the same output was turned on, and 5 minutes later the water level was 10 cm lower and the leak was repaired. How much time will it take for the two pumps to pump out the remaining water?

5.7. "Now it is too early for you to see this," said the teacher to his 33 pupils and ordered them, "Close your eyes." All the boys and half of the girls closed their right eye. All the girls and one third of the boys closed their left eye. How many of the pupils actually saw what they were not supposed to see?

5.8. Give an example of a fraction such that when the denominator is added both to the numerator and to the denominator its value is:

a) doubled; b) tripled; c) quadrupled.

5.9. Find all the fractions with denominator 15 that are greater than $\frac{5}{11}$ but less than $\frac{6}{11}$.

5.10. Find all the fractions with one-digit numerator and denominator that are greater than $\frac{7}{9}$ but less than $\frac{8}{9}$.

5.11. Which of the fractions $\frac{16}{20}$, $\frac{27}{36}$, $\frac{60}{72}$ is nearest to 1?

5.12. Compare the fractions $\frac{78}{79}$ and $\frac{90}{91}$.

5.13. Find an irreducible fraction that does not change its value if 4 is added to its numerator, and 10 is added to its denominator.

5.14. What number should be subtracted from the numerator of the fraction $\frac{537}{463}$ and added to its denominator so as to obtain $\frac{1}{9}$ after reduction?

5.15. Do there exist natural numbers m and n such that the three fractions $\frac{m}{n}$, $\frac{m+1}{n}$, and $\frac{m+1}{n+1}$ are irreducible?

5.16. Prove that the fraction $\frac{n(n+2)}{n+1}$ is irreducible.

5.17. Ivan Knownothing wants to arrange 2021 natural numbers in circle so that for each pair of neighbouring numbers the quotient under division of the greater number by the smaller one will be prime. Sophie Knoweverything says it is impossible. Is she right?

Operations with fractions

5.18. Pinocchio got on the train. Having covered half the distance, he fell asleep and slept until it remained to cover half the distance covered while he slept. What part of the route did he cover awake?

5.19. Peter spends $\frac{1}{3}$ of his time playing football, $\frac{1}{5}$, at school, $\frac{1}{6}$, watching movies, $\frac{1}{70}$, solving olympiad problems, and $\frac{1}{3}$, sleeping. Is such a life possible?

5.20. Basil borrowed his friend's book for three days. On the first day he read half the book, on the second day he read a third of the remaining pages, and on the third day he read the amount equal to half of what he had read in the first two days. Did he manage to read the whole book in three days?

5.21. Along a race track 12 flags are placed at equal distances. A sprinter running at constant speed covered the distance between the first and the eighth flag in 8 seconds. In how many seconds will he cover the distance between the first and the twelfth flag?

5.22. Two trains are running towards each other along parallel tracks, the first at 45 km/h, the second, at 36 km/h. A passenger sitting in the first train noticed that the second train passed him in 6 seconds. What is the length of the first train?

5.23. Place 7 barrels full of water, 7 barrels half filled with water, and 7 empty barrels in three trucks, so that each truck should contain the same number of barrels and the same amount of water.

5.24. The hand (pointer) of a balance is shifted. When a bunch of bananas was placed on the balance, the balance showed $1\frac{1}{2}$ kg. When a bigger bunch was placed on it, it showed $2\frac{1}{2}$ kg. When the two bundles were weighed together, it showed $3\frac{1}{2}$ kg. How much did the bananas actually weigh?

5.25. A tank was filled with water. This water was poured in equal amounts in three cans. It turned out that the first can was half full, in the second can, $\frac{2}{3}$ of its volume was filled, in the third, $\frac{3}{4}$ was filled. The volume of the tank and of each of the three cans is a whole number of litres. For what minimal volume of the tank was such a situation possible?

5.26. Is it possible to place inside a square of side 1 cm several non-overlapping squares such that the sum of lengths of their sides is 2000 cm?

5.27. How is it possible, without measuring instruments, to measure off 50 cm in a $\frac{2}{3}$ meter long string?

5.28. Using one teabag, one can brew two or three cups of tea. Molly and Tina divided a box of teabags evenly. Molly brewed 57 cups, Tina brewed 83 cups. Each of the teabags was used to brew 2 or 3 cups of tea. How many teabags were there in the box?

5.29. Three diggers dug three holes in two hours. How many holes will six diggers produce in five hours?

5.30. Three diggers dug three holes in three hours. How many holes will six diggers produce in five hours?

5.31. When Winnie-the-Pooh and Piglet divided a pie, Piglet complained that he got too little. Then Pooh gave him one third of his share. As the result, Piglet's share tripled. What were their shares initially?

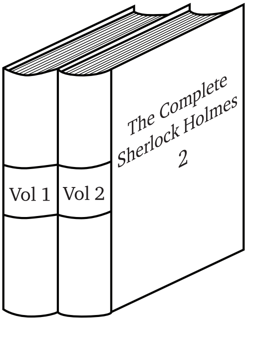
5.32. Father and son, working together, painted a fence in 12 hours. If the father were working alone, he would do the job in 21 hours. How long would it take the son to do this job alone?

5.33. A hiker left for a rally intending to walk for three days, covering a third of his route each day. He covered a third of his route the first day, but got tired and on the second day covered only one third of the remaining route. On the third day he covered only one third of the route that remained after the first two days. Then he still had 32 kilometres to go. What is the distance from the hiker's house to the rally?

5.34. Two boys bought a ball. The first boy contributed $\frac{1}{n}$ of the sum paid by the second boy. What part of the price was paid by the first boy?

5.35. Four friends are buying a boat. The first contributed half the sum paid by the others, the second, a third of the sum paid by the others, the third, one fourth of the sum paid by the others, while the fourth paid 1300 rubles. How much did the boat cost?

5.36. Twenty four trains run along the circular underground line. They move in the same direction at equal intervals. How many trains running at the same speed should be added so as to decrease the intervals by one fifth?
5.37. Calculate $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \frac{1}{32} - \frac{1}{64}$.
5.38. A team of mowers had to mow two fields, one of which was twice bigger than the other. Half a day the team worked on the bigger field, and during the second half of the day it divided into two equal groups. One group continued to work on the larger field, the other moved to the smaller one. By the end of the day, the larger field was entirely mowed, while in the smaller one a part remained, which was mowed during the next day by one mower. How many mowers were in the team?
5.39. Winnie-the-Pooh, Piglet, Rabbit, and Eeyore ate up a barrel of honey. Piglet ate half as much as Pooh, Rabbit, half of what Pooh did not eat, and Eeyore got only a tenth of the barrel. What part of the barrel did Rabbit get?
5.40. How can we equally divide seven apples between 12 boys if we are not allowed to cut any apple in more than four parts?
5.41. Two 500 page books, closely adjoining each other, stand on a shelf (see the figure). Each of the covers is 10 times thicker than a leaf of the paper on which both volumes are printed. There is a bookmark in each of the volumes. The distance between the bookmarks is one third of the total thickness of the two volumes. Between what pages of the second book is its bookmark if the bookmark of the first book is right in its middle?



5.42. In the Rhind papyrus (Ancient Egypt) among other facts one can see the representation of certain fractions as the sum of fractions with numerator 1, e.g.,

$$\frac{2}{73} = \frac{1}{60} + \frac{1}{219} + \frac{1}{292} + \frac{1}{x}.$$

Here we have replaced one of the denominators by x. Find x.

5.43. Several proper fractions are written in the following order: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, ... (fractions with denominator 2 in the increasing order, then fractions with denominator 3 in the increasing order, etc.; reducible fractions are *not* excluded).

a) How many fractions were written if the last one was $\frac{4}{65}$?

b) A total of 2016 fractions are written. Those which are less than $\frac{1}{2}$ are shown in red, the others, in blue. How many more red fractions than blue ones are there?

5.44. Find all the irreducible fractions whose value is doubled when 10 is added both to the numerator and the denominator.

5.45. a) Find three proper irreducible fractions whose sum is an integer and such that if all these fractions are "turned upside down" (i.e., replaced by its reciprocal), then the sum of the obtained fractions will also be an integer.

b) Same question, but the numerators should be distinct natural numbers.

5.46. In the relation

$$\frac{*}{*} + \frac{*}{*} + \frac{*}{*} + \frac{*}{*} = *,$$

is it possible to replace the stars by the digits 1 to 9 without repetitions so that it should be true?

5.47. The numerator and denominator of a fraction are natural numbers whose sum is 101. It is known that the fraction is less or equal to $\frac{1}{3}$. Find the largest possible value of such a fraction.

5.48. What is the greatest possible value of the expression

$$\cfrac{1}{a + \cfrac{2020}{b + \cfrac{1}{c}}},$$

where a, b, and c are pairwise distinct nonzero digits?

5.49. Arrange six distinct numbers in a circle so that each one equals the product of its two neighbours.

5.50. In each of two patches of land Grandpa planted the same number of swedes. If Granddaughter visits a patch, she gathers $\frac{1}{3}$ of the swedes, if the dog Zhuchka visits a path, it gathers $\frac{1}{7}$ of the swedes, and the mouse Myshka gathers only $\frac{1}{12}$ of the swedes. At the end of the week, 7 swedes were left in the first patch and 4, in the second. Did Zhuchka visit the second patch?

5.51. Introduce parentheses and signs of arithmetical operations in:

$$\frac{1}{2}\ \frac{1}{6}\ \frac{1}{6060} = 2020$$

so as to obtain a true identity.

Representation by fractions. Rebuses

5.52. Represent the fraction $\frac{1}{2}$ as the sum of two different fractions with numerator 1.

5.53. Represent the number 20 using four nines, the sign "+", and the notation for fractions.

5.54. Represent the number 55 using five fours, the sign "+", and the notation for fractions.

5.55. Represent the number 100 using six nines, the sign "+", and the notation for fractions.

5.56. Replace the stars by six distinct digits so that all the fractions will be irreducible and the resulting relation will be correct:

$$\frac{*}{*} + \frac{*}{*} = \frac{*}{*}.$$

5.57. Replace both stars by the same number so as to obtain a true relation:

$$\frac{20}{*} - \frac{*}{15} = \frac{20}{15}.$$

Mean value

The *arithmetic mean* of several numbers is the fraction whose numerator is the sum of these numbers and whose denominator is their quantity.

The average speed is the fraction whose numerator is equal to the length of the route covered, while the denominator is the time needed to cover the route.

5.58. We covered 12 km downriver at the speed of 12 km/h and then 12 km back upriver at the speed of 4 km/h. What was the mean speed of the boat?

5.59. The mean age of 11 members of a football team was 22. When one player was sent off the field, the mean age of the remaining 10 players became 21. How old was the the player who was sent off?

5.60. Professor Tester is conducting a series of tests on the basis of which he calculates the mean grade of each student. When John finished answering, he figured out that he would have earned the grade 90 had he gotten 97 for the last test. Had he gotten 73 for the last test, his mean grade would have been 87. How many tests did the professor conduct?

5.61. The arithmetic mean of four numbers is 10. If one of these numbers is crossed out, the arithmetic mean of the remaining three increases by 1; if one other number is crossed out instead, it increases by 2, while if a third number is crossed out instead, the arithmetic mean increases by 3. How will the arithmetic mean of the three other numbers change if the fourth number is crossed out?

5.62. Prove that if, to a collection of numbers, we add the number equal to their arithmetic mean, the arithmetic mean of the new collection will be equal to the arithmetic mean of the initial collection.

5.63. Fifty shooters took part in a competition. The first scored 60 points; the second, 80 points; the third, the arithmetic mean of the first two, the fourth, the arithmetic mean of the first three, and so on: each successive shooter scored the arithmetic mean of all the previous ones. How many points did the forty second shooter score?

5.64. While the ship was anchored in a port a sailor celebrated his 20th birthday. In that connection all six members of the crew gathered in the cabin.

— I am twice older than the cabin boy and six years older than the engineer, said the helmsman.

— And I am older than the cabin boy by as many years as I am younger than the engineer, besides, I am 4 years older than the sailor, said the boatswain.

— The mean age of the crew is 28, added the captain. How old is he?

5.65. Peter calculated the arithmetic mean of four numbers. If he deletes one of them, then the arithmetic mean of the remaining three will increase by 1; if deletes another one of the four, then the arithmetic mean of the remaining three will increase by 2; if he deletes a third one, then the arithmetic mean of the remaining three will increase by 3. How will the arithmetic mean change if Peter deletes the fourth number?

5.66. Seven distinct odd numbers are written on the blackboard. Tania calculated their arithmetic mean, Daniel rewrote them in increasing order and chose the middle one. If Daniel's number is subtracted from Tania's, the result will be $\frac{3}{7}$. Didn't one of them make a mistake?

Ratios and proportions

5.67. Six decorators will perform a job in 5 days. How many decorators will do it in 3 days?

5.68. Three decorators painted 60 windows in 5 days. How many windows will 5 decorators paint in 4 days?

5.69. Divide the number 88 into three parts proportional to $\frac{1}{2}, \frac{3}{4}, 1\frac{1}{2}$.

5.70. Divide the number 93 into three parts a, b, and c so that

$$a : b = 3 : 2 \quad \text{and} \quad b : c = 5 : 3.$$

5.71. Ivan had a certain amount of pastries; he ate some, then Tania came to visit him and they divided the remaining pastries equally. It turned out that Ivan ate five times more pastries than Tania. What part of the pastries had Ivan eaten before Tania arrived?

5.72. After Natasha had eaten half the peaches from a can, the level of the liquid in the can became one third lower. By what part of the new level will the liquid level decrease if Natasha eats half of the remaining peaches?

5.73. After Natasha had eaten one third of the peaches from a can, the level of liquid in the can became one fourth lower. By what part of the new level will the liquid level decrease if Natasha eats all the remaining peaches?

5.74. The father is 25 years older than his son. The ratio of the father's age to that of the son is $\frac{3}{2} : \frac{2}{3}$. How old are they?

5.75. Dolly and Tania live in different floors of the same stairwell of an apartment house, Dolly on the sixth. Leaving Dolly's flat, Tania, instead of walking down, as she should have done, started walking up. Having reached the top floor, she understood her mistake and started going down to her flat. It turned out that Tania had walked one and a half times more than if she had gone down directly. How many floors are there in the stairwell?

5.76. Getting up at 8:30 in the morning, a stoker fills the stove with coal to the brim. This requires 5 kg of coal. Each evening, going to bed (he always goes to bed at the same time of the evening), he fills the stove again, using exactly 7 kg of coal. When does he go to bed?

5.77. Gene took 10 cards with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 written on them, placed them in pairs on the table, and noticed that the ratio between the five resulting two-digit numbers was $1:2:3:4:5$. In the evening, when he wanted to show this interesting result to his father, he saw that the card with 0 on it was missing. Having thought a little, he used the remaining cards to form five numbers whose ratio was still $1:2:3:4:5$. How did he place the cards the first time and the second time?

Percentages

A *percent* is the $\frac{1}{100}$ part of a given quantity, n percent of a given quantity a equals $\frac{na}{100}$. Increasing a quantity by $n\,\%$ means multiplying it by $1+\frac{n}{100}$. Decreasing a quantity by $n\,\%$ means multiplying it by $1-\frac{n}{100}$.

5.78. The first book is 75 % cheaper than the second one. By what percent is the second book more expensive than the first?.

5.79. Potatatoes became 20 % cheaper. For the same amount of money, how much more potatoes can we buy?

5.80. Two students had the same scholarships. The A-student's scholarship was increased by 100 %, the B-student's, by 50 %. When the A-student got a B, his scholarship was lowered to the level of the B-student. By what percentage was his scholarship lowered?

5.81. In a high school all the students learn at least one ancient language, Greek or Latin; some learn both of these languages. We know that 85 % of the students learn Greek, 75 %, Latin. What percentage of the students learn both languages?

5.82. In a class, 50 % play basketball, 40 %, play tennis, 10 % play both games. What percentage of the students of that class play neither basketball, nor tennis?

5.83. For breakfast Johnny ate 40 % of a pie, Peter ate 150 gams. For lunch Dolly ate 30 % of the remaining pie and 120 gams more, and her cat Matilda licked up the rest, 90 gams of crumbs. How much did the pie weigh initially?

5.84. In a storeroom there were 100 kg of berries, 99 % of which consisted of water. After a long storage, the berries dried up a bit, so that the percentage of water became 98 %. How much do they weigh now?

5.85. The percentage of water in grass is 60 %, the percentage of water in hay is 20 %. How much hay can be obtained from 1000 kg of freshly mown grass?

5.86. Together, Peter and Johnny ate a pot of jam. Johnny ate 40 % less spoonfuls than Peter, but his spoon contained 150 % more jam than Peter's spoon. What fraction of the jam did Johnny eat?

5.87. 2 % of a positive number A is greater than 3 % of a positive number B. Is it true that 5 % of A is greater than 7 % of B?

5.88. Three pirates were sharing a sack of coins. The first took $\frac{3}{7}$ of the coins, the second, 51 % of the remaining coins, after which the third pirate was left with 8 coins less than the second. How many coins were in the sack?

5.89. Peter gave Johnny 111 chocolates. They ate some of them together, 45 % of the remaining chocolates were eaten by Johnny for lunch, and one third of the chocolates left after lunch were eaten by the cat Matilda. How many chocolates did she eat?

5.90. The fat issue of a newspaper costs 30 rubles, the thin issue is cheaper. The discount by the same percentage for all issues of newspapers has was established for pensioners; as the result, the thin issue now costs them 15 rubles. It is known that any one issue costs a whole number of rubles. What is the cost of the thin newspaper without discount and of the fat newspaper with discount?

5.91. Last year in School №1 the percentage of boys was 50 %, while in School №2, it was 80 %. This year the percentage of boys in each of the two schools did not change, but in the two schools together the percentage of boys has increased. Give an example showing how this could happen.

5.92. At the start of the year there were 25 pupils in a class. When 7 new pupils were added to the class, the percentage of straight-A students increased by 10 (if at the start of the year it was a %, now it is $(a+10)$ %). How many straight-A students are there in the class now?

Decimal fractions

A *decimal fraction* is a fraction with denominator 10, 100, 1000 etc., For instance, the fraction, $\frac{1}{10}$ can be written as 0.1, the fraction, $3\frac{27}{100}$ as 3.27; the fraction $10\frac{217}{1000}$, as 10.217.

5.93. Write the following numbers:

$$0.13;\ \frac{27}{200};\ 0.125.$$

in increasing order.

5.94. Write the following numbers:

$$\frac{3}{4};\ \frac{37}{50}; 0.7.$$

in increasing order.

5.95. Write the following numbers:

$$\frac{3}{7};\ \frac{6}{13};\ 0.4.$$

in increasing order.

5.96. What symbol can be placed between the digits 2 and 3 in order to obtain a number which is greater than 2 and less than 3?

5.97. Three cowboys entered a saloon. One bought 4 sandwiches, a cup of coffee and 10 doughnuts and paid 1 dollar 69 cents, the other bought 3 sandwiches, a cup of coffee and 7 doughnuts and paid 1 dollar 26 cents. How much did the third cowboy pay for a cup of coffee, one sandwich and one doughnut?

5.98. Basil multiplied some number by 10 and obtained a prime number. Peter multiplied the same number by 15, and also obtained a prime number. Is it possible that neither made a mistake?

5.99. Add a decimal point to each of the six numbers so as to obtain a true relation:

$$2016 + 2016 + 2016 + 2016 + 2016 = 46\,368.$$

5.100. Prove that a fraction whose denominator is a product of several 2's and 5's can be represented by a decimal fraction.

5.101. The denominator of an irreducible fraction is divisible by a prime number other than 2 or 5. Can such a fraction be represented as a decimal fraction?

Decimal fractions and percentages

Increasing a number by $a\,\%$ means multiplying it by $1 + \frac{a}{100}$, decreasing it by $a\,\%$ means multiplying it by $1 - \frac{a}{100}$. It is often convenient to write such fractions as decimal ones and then perform the required multiplication.

5.102. If a salary is first increased by 20 %, and then decreased by 20 %, will it decrease or increase?

5.103. In two stores milk was sold for the same price. Then in the first store it became 40 % cheaper, in the second one, first 20 % cheaper, and then 25 % cheaper. In which of the stores will it now be cheaper?

5.104. The length of a rectangular plot of land increased by 50 %, its width decreased by 10 %. How did the area change?

5.105. In spring Johnny lost 25 % of his weight, then in summer he gained 10 %, in autumn he lost 10 %, in winter he gained 20 %. Did he lose or gain weight during the year?

5.106. The total salary of all tsar's clerks was 1000 rubles per year, and all the clerks were equally paid. It was suggested to the tsar to decrease the number of clerks by 50 %, and increase the salary of the remaining clerks by 50 %. How will the total salary of the clerks change?

5.107. In two years a factory decreased its output by 51 %, and each year the decrease percentage was the same. What was this percentage?

5.108. Peter had 10 % more water in his bottle than Basil. Peter drank 11 % of the contents of his bottle, and Basil, 2 % of his. Who has more water left now?

5.109. There was an equal amount of water in two glasses.The amount of water in the first glass increased by 1 %, then by 2 %, then by 3 %, and so on up to 27 %. In the second glass, it was first increased by 27 %, then by 26 %, then by 25 %, and so on down to 1 %. Which of the glasses will contain more water?

5.110. Joe knows that in order to convert pounds to kilograms, one must divide the number of pounds by 2 and decrease the result by 10 %. From this Joe made the erroneous conclusion that to convert kilograms to pounds one must multiply the number of pounds by 2 and increase the result by 10 %. What will be the error percentage after such a conversion?

5.111. After each washing, the volume of a bar of soap decreases by 20 %. After how many washings will the soap lose more than half of its original volume?

5.112. Grandmasters and masters participate in a chess tournament. For what minimal number of participants can it happen that the grandmasters constitute less than a half, but more than 45 % of the total number of participants?

5.113. Pinocchio buried two bullions of gold and silver in the Field of Miracles. If the weather is fine, the gold bullion increases by 30 % a day and the silver bullion increases by 20 % a day. When the weather is poor, the gold bullion decreases by 30 % a day and the silver bullion decreases by 20 % a day. A week later, it turned out that one bullion increased, the other decreased. How many days was the weather fine?

5.114. Johnny comes to the shooting gallery having 100 dollars in his pocket. After each successful shot, the amount of money in his pocket automatically increases by 10 %, after each miss, it decreases by 10 %. Can it happen that after several shots, he has 80 dollars 19 cents in his pocket?

5.115. Alex, Boris, and Basil were gathering mushrooms. Boris gathered 20 % more mushrooms than Alex, but 20 % less than Basil. By what percentage more than Alex's was Basil's number of picked mushrooms?

5.116. Three pirates were sharing a chest of gold coins. The first took 30 % of all the coins, the second, 40 % of the remaining coins, the third, 50 % of what was left. After that 63 coins remained. How many coins were in the chest initially?

Aliquot fractions

An *aliquot fraction* is a fraction whose numerator is 1 and whose denominator is a natural number $n > 1$. For instance, the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{213}$ are aliquot.

5.117. Represent $\frac{1}{2}$ as the sum of two distinct aliquot fractions.

5.118. Represent $\frac{3}{4}$ as the sum of two aliquot fractions.

5.119. Divide three apples equally among four persons by cutting one apple into four parts and each of the other apples, into a lesser number of parts.

5.120. Represent $\frac{7}{8}$ as the sum of three aliquot fractions.

5.121. Divide 7 apples equally between eight people by cutting one apple into eight parts and each of the other six, into a lesser number of parts.

5.122. Represent 1 as the sum of three distinct aliquot fractions.

5.123. Represent 1 as the sum of four distinct aliquot fractions.

5.124. Represent the fraction $\frac{1}{3}$ as the sum of two distinct aliquot fractions.

5.125. Represent the fraction $\frac{1}{6}$ as the sum of two distinct aliquot fractions in four different ways (the order of summands is not taken into account).

5.126. Represent each of the the numbers $\frac{5}{8}$, $\frac{7}{10}$, $\frac{5}{6}$, $\frac{3}{5}$, $\frac{4}{5}$, and $\frac{2}{5}$ as the sum of distinct aliquot fractions.

5.127. In how many different ways (with the order of summands not taken into consideration) can the fraction $\frac{1}{25}$ be represented as the sum of two distinct aliquot fractions?

5.128. In how many different ways (with the order of summands not taken into consideration) can the fraction $\frac{1}{6}$ be represented as the sum of two distinct aliquot fractions?

5.129. In how many different ways (with the order of summands not taken into consideration) can the fraction $\frac{1}{12}$ be represented as the sum of two distinct aliquot fractions?

5.130. Represent in two different ways the fraction $\frac{3}{8}$ as the sum of two distinct aliquot fractions.

5.131. Can the number 1 be represented as the sum of the aliquot fractions $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$ with odd denominators?

5.132. Prove that any proper fraction can be represented as an aliquot fraction or the sum of several distinct aliquot fractions.

Chapter 6

Integers and Rational Numbers

Numbers that differ only by sign are called *opposite*.

The *modulus* of a number is the number itself when it is positive and the opposite number if it is negative. The modulus of zero is zero.

The modulus of the number a is denoted by $|a|$.

The modulus of a number is also called its *absolute value*.

Integers

Division with remainder is performed just like division with remainder of natural numbers. If a and b are integers and $b \neq 0$, then the division with remainder of a by b is the representation of the number a in the form $a = b \cdot q + r$, where the numbers q and r are integers and $0 \leq r < |b|$. The number r is the *remainder* of the division of a by b.

6.1. The product of 22 integers equals 1. Prove that their sum is not zero.

6.2. For what integers n is the number $\dfrac{n+9}{n+6}$ an integer?

6.3. The numbers 1 and -1 fill the cells of a 25×25 square table. For each row of the table, one calculated the product of numbers in this row. Prove that the sum of these products is nonzero.

6.4. A 2019×2021 table is filled with integers. It is known that the product of the numbers in each row is negative. Prove that there is a column such that the product of the numbers in it is also negative.

6.5. Find the largest value of the expression

$$aek - afh + bfg - bdk + cdh - ceg,$$

if each of the numbers $a, b, c, d, e, f, g, h, k$ can be equal to 1 or -1.
6.6. Introduce parentheses in the equation

$$1 - 2 \cdot 3 + 4 + 5 \cdot 6 \cdot 7 + 8 \cdot 9 = 1995.$$

so that it become true.
6.7. Prove that the number

$$2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2018 \cdot 2020 - 1 \cdot 3 \cdot 5 \cdot \ldots \cdot 2017 \cdot 2019$$

is divisible by 2021.
6.8. Prove that the number $53 \cdot 83 \cdot 109 + 40 \cdot 66 \cdot 96$ is composite.
6.9. On the Moon coins worth 1, 15, and 50 farthings are in use. Cyrano de Bergerac gave some coins for a purchase and received change; the number of coins in the change he received was one more than the number of coins he had given to the salesperson. What is the smallest possible price of his purchase?

Modulus of a number

6.10. Prove that if $ab < 0$, then $|a + b| < |a| + |b|$.
6.11. Are the following statements true:
a) if $|a| = |b|$ and $ab < 0$, then $a = b$; b) if $|a| = |b|$ and $ab > 0$, then $a = b$?
6.12. Find two different proper fractions, one with denominator 8, the other with denominator 13, such that the difference between the larger one and the smaller one is the smallest possible.
6.13. Find the smallest value of the expression $|36^k - 5^l|$, where k and l are natural numbers.

Infinite periodic fractions

An irreducible fraction whose denominator is divisible by a prime number other than 2 and 5 cannot be represented as a decimal fraction (Problem 5.101). Let us take such a fraction and try to perform long division of the numerator by the denominator. For instance, in the case if the fraction 1/3, we obtain 0.33333..., the digit 3 will repeat infinitely.

This can be clarified as follows. Let us divide 10 by 3 with remainder: $10 = 3 \cdot 3 + 1$. This equation implies that

$$\frac{1}{3} = \frac{10}{30} = 3 \cdot \frac{3}{30} + \frac{1}{30} = 0.3 + \frac{1}{30} = 0.33 + \frac{1}{300} = 0.333 + \frac{1}{3000} = \dots$$

Similar expressions with periodically repeated digits or groups of digits can be obtained from other fractions. For instance, $\frac{1}{11} = 09090909\dots$ or $\frac{5}{12} = 0.4166666\dots$

An expression in which a digit or a group of digits repeats periodically is known as an *infinite periodic fraction*. To indicate this repeating digit or group of digits one uses parentheses (round brackets):

$$\frac{1}{3} = 0.33333\dots = 0.(3), \quad \frac{1}{11} = 0.09090909\dots = 0.(09),$$
$$\frac{5}{12} = 0.4166666\dots = 0.41(6).$$

6.14. Represent the fraction 1/37 as an infinite periodic fraction.
6.15. Represent the fraction 1/7 as an infinite periodic fraction.
6.16. Represent the fractions 2/7, 3/7, 4/7, 5/7, 6/7 as infinite periodic fractions. Do you observe a pattern?
6.17. Represent the fraction 1/14 as an infinite periodic fraction. How is this representation related to the representation of the fraction 5/7?
6.18. Represent the fraction 1/17 as an infinite periodic fraction.
6.19. Represent the infinite periodic fractions 0.0(3) and 0.00(3) as proper fractions.
6.20. Show that $\frac{1}{9} = 0.(1)$, $\frac{1}{99} = 0.(01)$, $\frac{1}{999} = 0.(001)$, and so on.
6.21. Represent the numbers 0.(13) and 0.(238) as proper fractions.
6.22. Represent the numbers

a) $0.(3) \cdot 0.(4)$; b) $0.(12) + 0.(122)$

as infinite periodic fractions.
6.23. Compute he numbers 0.(23) and 0.(2323) as in the solution of Problem 6.21 and verify that these numbers are equal.
6.24. Verify that all the numbers 0.(9), 0.(99), 0.(999), and so on are equal to 1.
6.25. Represent the number $1 - 0.(85)$ as a proper fraction and as an infinite periodic fraction.

Positive and negative sums

6.26. Place five integers in a row so that the sum of any two neighboring numbers be negative, but the total sum of the five numbers be positive.

6.27. Place five integers in a row so that the sum of any three neighbouring numbers be negative, but the sum of all five be positive.

6.28. Place five integers in a row so that the sum of any four neighbouring numbers be negative, but the sum of all five be positive.

6.29. Place seven integers in a row so that the sum of any two neighbouring numbers be negative, but the sum of all seven be positive.

6.30. Place seven integers in a row so that the sum of any three neighbouring numbers be negative, but the sum of all seven be positive.

6.31. Is it possible to place six integers in a row so that the sum of any two neighbouring numbers be negative, but the sum of all six be positive?

6.32. Is it possible to place nine integers in a row so that the sum of any three neighbouring numbers be positive, but the sum of all nine be negative?

6.33. Several integers are placed in a row so that the sum of any three neighbouring numbers is positive. Can we assert that the sum of all the numbers is positive if there are a) 18; b) 19; c) 20 numbers?

6.34. Is it possible to fill a 5×5 table with integers so that the sum of numbers in any row be positive, while the sum of numbers is any column will be be negative?

6.35. Is it possible to fill a rectangular table with 3 rows and 4 columns with integers so that

a) the sum in each row be -20, the sum in each column be -15;

b) the sum in each row be -20, the sum in each column be -16?

6.36. Place 9 numbers in a row so that the sum of any 4 neighbouring numbers be positive, but the sum of any 7 neighbouring numbers be negative.

6.37. Place 16 numbers in a row so that the sum of any 7 neighbouring numbers be negative, but the sum of any 11 neighbouring numbers be positive.

Chapter 7

Equations

Constructing equations

7.1. The teacher asked Pinocchio to multiply a number by 4 and add 15 to the result, but Pinocchio first multiplied the number by 15 and then added 4; nevertheless, the answer turned out to be correct. What was that number?

7.2. In a certain American firm each employee was either a Democrat or a Republican. Initially there were as many Democrats as Republicans. Then 3 Republicans became Democrats, and after that there were twice as many Democrats as Republicans. How many employees are there in the firm?

7.3. One positive number was increased by 1%, the other, by 4%. Could their sum increase by 3%?

7.4. According to the contract, Johnny is paid 48 dollars for each workday, and for each missed workday, 12 dollars are deducted from his salary. In 30 days Johnny found out that he has not earned anything, but he does not owe anything to his employer, either. How many days did he actually work during this 30 day period?

7.5. One has three boxes of apples. The first box contains 6 kg of apples less than the two other boxes together, while the second one contains 10 kg less than the two other boxes together. How many kilograms of apples are there in the third box?

7.6. John and Mary were decorating a Christmas tree. In order to avoid a possible conflict, their mother indicated which branches each of them must decorate and which toys they should use to do that; the number of toys and the number of branches were the same for both of them. John tried to hang up one toy on each branch, but he lacked one branch to do it; Mary tried to hang up two toys on each branch, but one branch remained empty. How many branches and toys did the mother assign to each of the children?

7.7. At a certain moment in a chessgame between Johnny and Freddy, Johnny had half as many pieces as Freddy, and five times pieces less than the number of free squares on the chessboard. How many of Freddy's pieces had been taken by that time?

7.8. George, Mary, and Ilona are decorating a Christmas tree. It is known that Mary hung up twice as many toys as George, George hung up 15 toys less than Ilona, while Ilona hung up twice as many toys as George and Mary taken together. How many toys are on the tree?

7.9. Balance scales with unequal arms achieve equilibrium when the ratio of masses on two plates equals the inverse of the ratio of lengths of corresponding arms. When Johnny weighed a sack of flour on one plate, its apparent weight was 50 kg, and when he weighed the same sack on the other plate, the apparent weight was 32 kg. How much does the sack actually weigh?

7.10. One kilogram of beef with bones costs 78 rubles, one kilogram of beef without bones costs 90 rubles, and one kilogram of bones costs 15 rubles. What is the weight of bones contained in one kilogram of beef with bones?

7.11. Place numbers at the vertices of a pentagon so that the sum of the numbers at the endpoints of a certain side be equal to 1, the sum of the numbers at the endpoints of the next side be equal to 2, . . . , at the endpoints of the fifth side, be equal to 5.

7.12. The King told the Queen: "At present I am twice as old as you were when I was as old as you are today. When you will be as old as I am today, the sum of our ages will be 63". What are the ages of the King and the Queen?

Motion

7.13. A truck covers a certain distance in 10 hours. If it could cover 10 km more in an hour, then the same distance would be covered in 8 hours. What is the speed of the truck?

7.14. Two cars started simultaneously from the cities A and B towards each other and met after 8 hours. If the speed of the "A-car" had been greater by 14% and the speed of "B-car" had been greater by 15%, then the cars would have met in 7 hours. The speed of which car is greater? Find the ratio of their actual speeds.

7.15. Two cars started simultaneously from the cities A and B towards each other. In 7 hours the distance between them was 136 km. Find the distance between A and B if one car covers the whole distance in 10 hours and the other one covers the whole distance in 12 hours.

7.16. Having covered half of its trajectory, a motorboat increased its speed by 25% and arrived one half hour early. How long was it moving?

7.17. What was the angle between the hour and the minute hand of a clock if 20 minutes later they formed the same angle?

7.18. A plane flew out of the city A at noon and landed at the city B at 2 pm local time. At midnight it flew back to A and landed there at 6 am local time. How long did the flights take?

7.19. Yesterday I walked on skis 3 kilometers less than the day before yesterday, and 40 kilometres less than today and the day before yesterday taken together. How many kilometres did I cover today?

7.20. Winnie the Pooh and Piglet simultaneously started out to visit each other, but because both of them were watching the jackdaws flying over them, neither one of them noticed the other when they met. After they met, Piglet reached Pooh's house in 4 minutes, while Pooh made it to Piglet's house in 1 minute. In how many minutes after they left did they meet?

7.21. Two trucks left from points A and B towards each other, reached the opposite points and turned back. The first time they met at the distance of 60 km from A, the second time, 80 km from B. Find the distance between A and B.

Reverse engineering

7.22. Lotuses grow in a pond. In a day (24 hours) each lotus divides into two, so that each lotus is replaced by two of the same size. A day later, each of the new lotuses divides into two, and so on. Thirty days after the introduction of the first lotuses, the pond is entirely covered by lotuses. How much time after the start was the pond half covered by lotuses?

7.23. The whole household is trying to pull out a big turnip. Grandpa is 2 times stronger than Grandma, Grandma is 4 times stronger than Granddaughter, Granddaughter is 5 times stronger than the dog

Zhuchka, the dog is 6 times stronger than Kitty Cat, who is 6 times stronger than Mousey Mouse. Grandpa, Grandma, Granddaughter, Zhuchka, Kitty Cat with the help of Mousey can pull out the turnip, but without Mousey they can't. How many mice should be called so that they will succeed in pulling out the turnip?

7.24. Mother placed some plums on the table and told the children to divide them evenly when they come back from school. Anna was the first to return, and she took one third of the plums. Then Boris returned, he took one third of the remaining plums and went away. Then Victor came and took four plums — one third of the plums that he saw. How many plums did Mother leave?

7.25. Mother divided apples between her three sons. To the first, she gave half of all the apples and half an apple more, to the second she gave half of the remainder plus half an apple, to the third, half of the new remainder and the remaining half apple. How many apples did each of the sons receive?

7.26. There were some lemons in a box. At first, half of all these lemons plus half of one lemon were taken out of the box, then half of the remainder plus half of one lemon was taken out. After that, 31 lemons remained in the box. How many lemons were in the box initially?

7.27. Three sisters divided plums in the following way: the first took one third of all the plums, plus 8 plums, the second, a third of the remainder, plus 8 plums, the third took one third of the new remainder plus the remaining 8 plums. How many plums did each sister get?

7.28. There were matches in a box. Their number was doubled and then 8 matches were removed. The remaining matches were doubled again and then 8 matches were removed again. When the same operation was performed a third time, the box was empty. How many matches were there initially?

7.29. The devil told a lazy man, "Each time you cross this magic bridge, the amount of money you have will double. For that, each time you cross the bridge, you must give me 24 dollars." The lazy man crossed the bridge three times — and it turned out that he had no money left (i.e., the third time he gave to the devil all the money that he had at that moment). How much money did the man have initially?

7.30. Two pirates were playing for gold coins. First the first one lost half of his coins (and gave them to the second one), then the second lost half of his coins, then the first one lost half of his coins once more. As the result, the first had 15 coins, the second, 33. How many coins did the first pirate have before the game began?

7.31. Solve the equation

$$1993 = 1 + 8 : (1 + 8 : (1 - 8 : (1 + 4 : (1 - 4 : (1 - 8 : x))))).$$

7.32. From a positive integer, the sum of its digits was subtracted. The same was done with the obtained number and so on. After 11 such subtractions, zero was obtained for the first time. What was the initial number?

7.33. Four ones and five zeroes are placed in a circle. Every second the following operation is performed with these numbers: 0 is placed between two neighbouring numbers if they differ, and 1, if they are equal; after that, the old numbers are erased. Can it happen that, after several such operations, all the numbers will become equal?

Equations in integers

In some situations it is necessary to to find not all the solutions of an equation, but only those that are whole numbers.

7.34. Find a two-digit number that is equal to the doubled sum of its digits.

7.35. Find a two-digit number that is equal to twice the product of its digits.

7.36. Two fishermen caught 70 fish, and 5/9 of the first one's catch were carps, and 7/17 of the other fisherman's catch were perch. How many fish did each of them catch?

7.37. Two fishermen caught 80 fish, 5/9 of the first one's catch were carps, and 7/11 of the other fisherman's catch were perch. How many fish did each of them catch?

7.38. Balance scales were in equilibrium when on one plate only 2 kg weights were placed and only 5 kg weights were on the other plate, 14 weights in all. How many 2 kg weights were used?

7.39. A store acquired 223 litres of olive oil in cans of 10 and 17 litres. How many cans were there?

7.40. In a basket there are 20 fruit: apples, pears, and peaches. How many apples are there in the basket if there are 9 times as many peaches as pears?

7.41. Find a three-digit number ABB such that the product $A \cdot B \cdot B$ equals the two-digit number AC and the product $A \cdot C$ equals C.

7.42. Children were eating candies. Each ate 7 candies less than all the others taken together, but more than one candy. How many candies were eaten?
7.43. Each member of a family drank a cup of coffee with milk. In particular, Katya had 1/4 of all the milk and 1/6 of all the coffee. How many people are there in the family?
7.44. If one adds 16 to 20, the result is 36, which is a perfect square. If one subtracts 16 from the same number, the result is 4, which is also a perfect square. Do there exist other whole numbers which become perfect squares when 16 is added to them as well as when 16 is subtracted? How many are there?
7.45. Find 2 two-digit prime numbers that are obtained from each other by interchanging digits and whose difference is a perfect square.
7.46. Johnny, Timmy, and Billy were throwing snowballs at each other. Johnny threw the first snowball. Then, after any snowball hit one of them, if it was Timmy, he would throw 6 snowballs, if it was Billy, he would throw 5, and if it was Johnny, 4. After some time, the game ended. Find out how many times each one was hit if the total number of snowballs that missed was 13. (No one threw any snowballs at oneself, and no single snowball could hit two people.)
7.47. On the 16th of February 2020 a boy said: "The difference between the number of (complete) months and (complete) years that I have lived became equal to 111 for the first time today". When was he born?
7.48. In the big antigreed pill there is 11 g of antimatter, in the middle-sized one, 1.1 g, in the small one, 0.11 g. The doctor prescribed greedy Billy to swallow 20.13 grams of anti-matter. Will Billy be able to fulfil the doctor's prescription using at least one of each of the types of pills?
7.49. A shepherd was grazing a 100-head herd. He was paid 200 rubles. For each bull he was paid 20 rubles, for a cow, 10 rubles, for a calf, 1 ruble. How many bulls, how many cows, and how many calves were there in the herd?
7.50. Prove that the equation

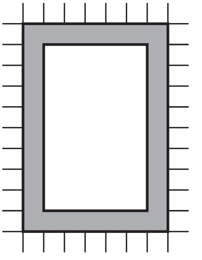
$$x^2 = 14 + y^2$$

has no natural solutions.
7.51. A huge orchestra was demonstrating its art outdoors. First the musicians were lined up to form a square, then they realigned to form a rectangle so that the number of files increased by 5. How many musicians were in the orchestra?

7.52. Solve the equation

$$2x + 5y = xy - 1$$

for integer x and y.



7.53. Find all the rectangles whose sides are natural numbers and the value of whose perimeter equals the value of their area.

7.54. George drew a rectangle on a square-lined sheet of paper (along the lines), see the figure. After that he drew a frame around it of one-square width. It turned out that the area of the rectangle and the area of the frame are equal. What was the size of the original rectangle?

7.55. What can be the last digit of the square of a natural number if the next-to-last digit is odd?

7.56. A three-digit number begins with the digit 7. Another three-digit number is obtained from it by placing that digit at the end of the number. The obtained number is smaller by 117 than the initial one. What number was obtained?

7.57. In the year x^2, my nephew will be x years old. In what year was he born?

7.58. Four little girls, Anya, Katya, Liza, and Dasha, and four little boys, Kolya, Petya, Tolya, and Vassya, shared 32 peaches among themselves as follows. Anya got 1 peach, Katya got 2, Liza, 3, and Dasha, 4. Each of the four boys is the brother of exactly one of the four girls. Kolya took as many peaches as his sister, Petya, twice as many as his sister, Tolya, three times as many as his sister, and, finally, Vassya, four time as many as his sister. For each of the girls, find the name of her brother.

7.59. A kangaroo jumps along a straight line. Pushing off its left foot, it jumps 3 m, off its right foot, 5 m, off both feet, 7 m. How can it cover exactly 200 m in 30 jumps if it is allowed to jump only in one direction?

7.60. In how many ways can an odd prime number p be represented as the difference of two squares?

7.61. Solve the equation

$$3x^2 + 5y^2 = 345$$

in natural numbers.

7.62. Solve the equation

$$1 + x + x^2 + x^3 = 2^y$$

in natural numbers.

7.63. Solve the equation

$$pqr = 7(p + q + r)$$

in prime numbers.

7.64. Find all the natural numbers n for which $2^n + 33$ is a perfect square.

7.65. Is it possible to find 10 consecutive natural numbers such that the sum of their squares is equal to the sum of squares of the next 9 consecutive natural numbers?

7.66. Prove that the equation $4^k - 4^l = 10^n$ has no solutions in whole nonnegative numbers.

7.67. Solve the equation

$$x^3 + 3 = 4y(y + 1)$$

in natural numbers.

7.68. Find the integer solutions of the equation

$$x^2 + y^2 + z^2 = 4(xy + yz + zx).$$

7.69. Solve the equation $3^n + 55 = m^2$ in natural numbers.

7.70. The product of two numbers 2.75 and 8 is equal to the sum of the digits that constitute them: $2.75 \cdot 8 = 22 = 2 + 7 + 5 + 8$. Find at least one other such pair of numbers.

7.71. Find all the natural numbers that are 13 times greater than the sum of their digits.

7.72. Three stores received 1990 books. During the first three days, the first store sold $\frac{1}{37}$, $\frac{1}{11}$, and $\frac{1}{2}$ of the obtained books, the second store, $\frac{1}{19}$, $\frac{1}{9}$, and $\frac{1}{3}$ books, the third, $\frac{1}{25}$, $\frac{1}{30}$, and $\frac{1}{10}$. How many books did each store receive?

7.73. There is more than one child in a family, and in this family each child has brothers. When each child was asked how many brothers he or she had, the sum of all the numbers given in their answers turned out to be 35. How many children are there in the family?

7.74. Find all the natural numbers n and m for which $m^{n+m} = n^{12}$ and $n^{n+m} = m^3$.

Chapter 8

Inequalities

Comparison of numbers

8.1. Which of these numbers is greater: $1-2+3-4+5-\ldots+99-100$ or $1+2-3+4-5+6-\ldots-99+100$?

8.2. Find a fraction with denominator 16 which is greater than 11/17, but less than 12/17.

8.3. Find a fraction with denominator 26 which is greater than 8/25, but less than 9/25.

8.4. A certain salary was first decreased by $a\,\%$, and then increased by $b\,\%$. As a result, the initial value of the salary was restored. Prove that $b > a$.

8.5. Dorothy prepared 30 cakes to treat the Scarecrow, the Tin Man, the Cowardly Lion, and the Wizard. After a while it turned out that the Scarecrow and the Tin Man together ate as many cakes as the Lion and the Wizard, while the Scarecrow and the Lion, 6 times more than the Tin Man and the Wizard. How many cakes did each one of them eat, if it is known that the Wizard ate less than each of the others? (Each of them ate each cake entirely, and each ate at least one cake.)

8.6. There are 27 pupils in a class, but not all of them attended the Physical Education lesson. The teacher divided those present into two equal teams to play a game. The first team contained half of the boys present and one third of the girls present, while the second team contained half of the girls present and one fourth of the boys present. The remaining pupils present helped the umpire. How many helpers did he have?

8.7. During a minute of rest, the musketeers (Athos, Porthos, Aramis, and d'Artagnan) decided to compare their strengths in a tug of war. Porthos and d'Artagnan easily defeated Athos with Aramis, but when Porthos joined forces with Athos, they did win, but with difficulty. When Porthos with Aramis pulled against the other two, neither pair could win. Can you rank the musketeers in order of force?

8.8. Rank in decreasing order the positive numbers a, b, c, d about which it is known that $a > b + c$, $a + b = c + d$ and $b + d > a + c$.

8.9. A group of children are holding flags in their hands. There are five times less children with an equal number of flags in the two hands than those with a different number. When all the children switched one flag from one hand to the other, the number of those who now had the same number of flags in the two hands became twice less than those who had a different number. Is it possible that initially more than one half of all the children had exactly one flag more in one hand than in the other?

8.10. In the equation $\frac{1}{x} = \frac{1}{y} + \frac{1}{z}$, can one of the numbers x, y, or z be a one-digit number, another, a two-digit number, the third, a three-digit number?

8.11. The numbers a, b, c are natural and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{41}{42}.$$

8.12. A pharmacist has three weights that he used to weigh 100 g of a drug, 101 g of another drug, and 102 g of yet another drug. He always placed the weights on one plate of the scales, the merchandise on the other. Could it happen that every weight is of mass less than 90 g?

8.13. Prove that the example in Problem 8.12 is the only one possible.

8.14. There was not enough room to assemble a model house. The architect changed the design, removing two entrances and adding three floors; as a result, the number of flats increased. Then the architect decided to remove two more entrances and add three more floors. Could it be that there were less flats in this third project than in the original one? (The number of floors at each entrance and the number of flats at each floor must be the same.)

8.15. Sasha invited Petya and told him that he lives in a ten-entrance building in the tenth entrance in the flat number 333, but forgot to indicate the floor number. Coming to the building, Petya noticed that it has nine floors. To what floor must he go? (The number of flats on each floor is the same, the flats are numbered consecutively in all the building, starting with 1).

8.16. Which number is greater, 2^{300} or 3^{200}?

8.17. Which number is greater, 31^{11} or 17^{14}?

8.18. Which number is greater, $\frac{10^{10}+1}{10^{11}+1}$ or $\frac{10^{11}+1}{10^{12}+1}$?

8.19. Find all the pairs of prime numbers p and q for which $7p+1$ is divisible by q, while $7q+1$ is divisible by p.

Maximal and minimal numbers

8.20. An electronic watch shows hours and minutes. Practising his computational skills, Pinocchio calculates the sum of digits appearing on the watch (for example, when the watch shows 16:15, the sum is $1+6+1+5=13$). Find the time of the day when this sum is maximal.

8.21. Thirty pikes were thrown into a pond and started to eat each other up. A pike is satiated if it has eaten no less than 3 other pikes (it does not matter whether the fish it had eaten were satiated or not). What is the maximal number of pikes that were satiated at least once during their life in this pond?

8.22. The product of one billion (10^9) natural numbers equals one billion. What is the maximal possible value of the sum of these numbers?

8.23. It is known that $a > b$ and $c > d$. Which of the two numbers $(a+c)(b+d)$ and $(a+d)(b+c)$ is greater?

8.24. Using each of the digits from 1 to 9 once, construct three three-digit numbers whose product is maximal.

8.25. The numbers 415, 43, 7, 8, 74, 3 are written on cards. Place the cards in a row so as to minimise the obtained ten-digit number.

8.26. A hare bought seven drums of different sizes and seven pairs of sticks (also of different sizes) for her seven bunnies. A bunny that sees that their drum is bigger and their sticks are longer than those of one of their siblings, starts drumming. What maximal number of bunnies can start drumming?

8.27. Ten people came for a visit in galoshes. They left one by one, and each of them would put on an arbitrary pair of galoshes among those that they could put on (i.e., of size no less than their own). What maximal number of visitors did not succeed in putting on any galoshes?

8.28. The government of Badluck Island (which has 96 inhabitants) decided to implement five reforms. Each reform dissatisfies exactly half of the population. A citizen will participate is a meeting of protest if he/she is dissatisfied with more than half of the reforms. What maximal number of people can the government expect at the meeting?

8.29. Find the least natural number which ends in 56, is divisible by 56, and whose sum of digits is equal to 56.

8.30. Johnny looked at a 24-hour electronic clock that shows time in hours and minutes, and noticed that all the four digits were different. When he looked at the clock again, he saw four other different digits. (All the eight digits that he had seen were different.) What is the least time that could have passed between these two observations?

8.31. Aladdin came to the cave of wonders containing gold and diamonds and a chest that could be used to carry the treasure out. The chest filled with gold weighs 200 kg, the chest filled with diamonds weighs 40 kg, the empty chest weighs nothing. A kilogram of gold can be sold for 20 dinars, a kilogram of diamonds for 60 dinars. Aladdin can lift and carry no more than 100 kg. What maximal sum of money can he get for the treasure after one trip to the cave?

8.32. A monkey is happy when it has eaten 3 different fruits. What maximal number of monkeys can become happy if there are 20 pears, 30 bananas, 40 peaches, and 50 tangerines?

8.33. Ten workers are to produce 50 items. Each item must first be painted and then put together. Painting takes 10 minutes, putting together, 20 minutes. After being painted, the items must dry for 5 minutes. How can one divide the workers into painters and fitters so that the task be completed in the shortest time?

Deleting digits

8.34. From the number

$$1234512345123451234512345$$

delete 10 digits so that the resulting number be maximal.

8.35. From the number

$$1234567\ldots5657585960$$

(the first 60 natural numbers written in a row without spaces) delete 100 digits so that the resulting number be:

a) minimal (the number can begin with zeros);

b) minimal (the number cannot begin with zeros);

c) maximal.

8.36. Write the first 10 prime numbers in a row (without spaces) and then delete 6 digits so as to get the greatest possible number.

Order and ordering

8.37. Dorothy, the Scarecrow, the Tin Man, and the Cowardly Lion are sitting on a bench. If the Tin Man, sitting to the right of the others, will move and sit between Dorothy and the Scarecrow, then the Scarecrow will be at the left extremity of the bench. In what order did they sit initially?

8.38. Together ten people gathered 46 mushrooms. No two of them gathered the same number of mushrooms. How many mushrooms did each of them gather?

8.39. Four girls came to the skating rink, each with her brother. They divided into pairs and started skating. It turned out that in each pair, the male partner was taller than the female partner and no girl skated with her brother. The tallest in the group was Yura Vorobiev, the next highest was Andrey Egorov, then came Lusya Egorov, then Seriozha Petrov, Olya Petrov, Dima Krymov, Inna Krymov, and Anya Vorobiev. Determine who skated with whom. Andrey, Dima, Seriozha, and Yura are male names; Anya, Inna, Lyusya, and Olya are female names. Each pair of siblings shares one family name.

8.40. Two hundred people are lined up in a rectangle with 10 of them in each rank and 20 in each file. The tallest person in each file is chosen, and then among these 10 persons, the shortest is distinguished. Besides, in each rank, the shortest person is chosen, and among these 20 chosen persons the tallest is distinguished. Which one of these two distinguished persons is taller?

8.41. If the "tomorrow" for yesterday was Thursday, then what day of the week will be "yesterday" for the day after tomorrow?

8.42. If the puzzle that you solved before you solved this one was more difficult than the puzzle you solved after you solved the puzzle that you solved before you solved this one, then was the puzzle that you solved before solving this one more difficult than this one?

8.43. A collection of weights has the following properties:

a) it contains 5 weights of different masses;

b) for any two weights, we can find two other weights of the same total mass as the given ones.

What is the least number of weights in such a collection?

8.44. Sixth form pupils are lined up in a row, and before this row of sixth form pupils, a row of fifth form pupils stands, with the same number of people. Each fifth form pupil is shorter than the sixth form pupil standing behind him. Show that if both the rows are rearranged by height, then each fifth form pupil will still be shorter than the sixth form pupil standing behind him.

Chapter 9

Logic, Combinatorics, Sets

Confused inscriptions

9.1. Three sacks contain sugar, wheat, and flour. “Wheat” is written on the first sack, “flour” on the second one, and “wheat or sugar” on the third. What do the sacks contain if all the inscriptions are wrong?

9.2. The red can has the inscription “sugar”, the blue can, the inscription “the green can is empty”, the green can, “flour”. It is known that one can contains sugar, one can contains flour, and one can is empty. Which can contains sugar if it is known that all the inscriptions are wrong?

9.3. The stepmother, leaving for the ball, gave Cinderella a sack containing a mixture of wheat and poppyseed, and demanded to separate them out. When Cinderella left the house, the stepmother left three sacks: one contained wheat, the other, poppyseed, the third one, the unseparated mixture. In order to avoid confusing the sacks, Cinderella attached tags to them with the inscriptions “poppyseed”, “wheat”, and “mixture”.

The stepmother returned from the ball first and attached the tags so that all the inscriptions were wrong. One of the assistants of the Fairy warned Cinderella that all the inscriptions were incorrect. Then Cinderella took out a single grain from one of the sacks and immediately guessed what are the contents of all the sacks. How did she do that?

Somebody told the truth, somebody lied

9.4. On Dorothy's birthday the Scarecrow wants to find out her age. The Tin Man says that Dorothy's age is greater than 11, the Cowardly Lion says that it is greater than 10. How old is Dorothy if it is known that, of the Tin Man and the Lion, exactly one is wrong?

9.5. The Tsar learned that one of the three strong men, Muromets, Nikitich, and Popovich, killed the dragon Gorynych. The Tsar summoned them to court.

Quoth Muromets, "Nikitich killed the dragon."

Quoth Nikitich, "Popovich killed the dragon."

Quoth Popovich, "I killed the dragon."

It is known that only one told the truth, the other two lied. Who killed the dragon?

9.6. A bottle of Jamaican rum disappeared from the captain's cabin of a pirate boat. Three pirates were suspected: Harry, Tom, and One-eyed Charlie. Harry said, "I did not touch your lousy rum, neither did Tom." Tom said, "I swear by God, sir, Harry is innocent. It was the One-eyed who pilfered the rum." Charlie said, "Your bottle was taken by Harry. I have nothing to do with it." The captain succeeded in finding out who stole the rum. It turned out that one of the suspects lied twice, one told the truth twice, while the third one lied once and told the truth once. Also the thief acted on his own. Who was he?

9.7. One of five brothers baked a cake for his mother.

Nikita said, "It was Gleb or Igor."

Gleb said, "It was neither me, nor Dima."

Igor said, "Both of you are joking."

Andrey said, "No, one of them told the truth, the other didn't."

Dima said, "Andrey, you're wrong."

The mother knows that three of her sons always tell the truth. Who baked the cake?

9.8. Four students were discussing the answer to a problem.

John said, "It's the number 9."

Peter said, "It is a prime number."

Susie said, "It is an even number."

Mary said, "The number is divisible by 15."

One boy and one girl answered correctly, the other two were wrong. What is the answer to the problem?

9.9. One boy said, "Petya has more than 10 books."

Another said, "Petya has less than 10 books."

The third said, "He surely has at least one book."

It turned out that only one of these assertions is true. How many books does Peter have?

9.10. The inhabitants of the city A always tell the truth, those of the city B always lie, those of the city C, alternatively tell the truth or lie (having lied, next time they tell the truth, having told the truth, next time they lie). The fire station that services all three cities received a telephone call:

"We have a fire!"

"In what city?"

"In the city C."

To which city should the firemen go if the fire actually occurred?

9.11. Some ladybugs gathered in a clearing. If a given ladybug has 6 dots on her back, then she always tells the truth, if she has four dots, it always lies. There are no other ladybugs in the clearing.

One ladybug said, "We all have the same number of dots."

The second one said, "All together we have 30 dots on our backs."

"No, all together we have 26 dots on our backs," objected the third.

"Of these three only one told the truth," said each of the others.

How many ladybugs were there in the clearing?

9.12. Thirteen children sat around a round table and agreed that each boy would always lie to girls and tell the truth to other boys, whereas the girls would always lie to boys and tell the truth to other girls. One of the children told his/her right neighbour,

"Most of us are boys."

The latter told the next child to the right,

"Most of us are girls."

And so on, alternating these two assertions, until the last child told the first,

"Most of us are boys."

How many boys were there at the table?

9.13. Three parrots, Gesha, Kesha, and Roma, got together. One of them is honest, he always tells the truth, another one is a liar, he always lies. The third one is clever: sometimes he tells the truth, sometimes he lies. To the question

"Who is Kesha?"

the parrots answered as follows:

Gosha: "Kesha is a liar".

Kesha: “I am the clever one”.

Roma: “He is an absolutely honest parrot”.

Which one of the parrots is honest, which one is a liar, which one is the clever parrot?

9.14. In Emptyland there are three tribes: elves, goblins, and hobbits. Elves always tell the truth, goblins always lie, while hobbits alternatively lie and tell the truth. One day, at a round table, several inhabitants of Emptyland were feasting. One of them, pointing to his neighbour on the left, said, “He is a hobbit.”

The mentioned neighbour said,

“My neighbour to the right lied.”

Then the left neighbour of this neighbour repeated the same phrase, then their neighbour repeated it, and so on: several times in a round each of these creatures repeated that their right neighbor had lied. Determine from what tribes did these creatures come if there were:

a) nine;

b) ten inhabitants of Emptyland at the round table.

9.15. It is known that Jackal always lies, Lion always tells the truth, Parrot always repeats the previous answer he heard (and if it is asked first, it randomly answers “Yes” or “No”), Giraffe gives only honest answers, but to the previous question it was asked instead of the current one (and if asked first, it answers arbitrarily). In a fog, the wise Hedgehog met Jackal, Lyon, Parrot, and Giraffe and decided to find out in what order they were standing. Having asked all of them, from left to right, “Are you Jackal?”, he only found out where Giraffe was standing. Having asked all of them (in the same order) “Are you Giraffe?”, he only found out where Jackal was standing, but it was not yet clear to him where the three remaining animals were. Only when the first animal answered “Yes” to the question “Are you Parrot?”, it became clear to Hedgehog in what order the animals stood. What was the order?

Knights and knaves

In the problems about knights and knaves, the *knights* always tell the truth and the *knaves* always lie. They live on Knight-Knave Island and look exactly the same. This island is sometimes visited by *tourists*, who sometimes tell the truth, and sometimes lie.

9.16. On Knight-Knave Island a boy said that he is a knave. Does he live in the island or is he a tourist?

9.17. A traveller who visited the Knight-Knave Island asked his guide to find out whether a person that they met was a knight or a knave. The guide said that this person claimed that he was a knight. Who was the guide, a knight or a knave?

9.18. What question could one ask of an inhabitant of the island in order to find out whether he is a knight or a knave?

9.19. To what question do the knights and the knaves answer in the same way?

9.20. What question can be asked of a knight twice and get different answers?

9.21. A tourist met two boys living on the Knight-Knave Island, Johnny and Freddy. Freddy said that at least one of them is a knave. Who are these boys?

9.22. Each of the inhabitants of the Knight-Knave Island that gathered in a square, said, "All of you are knaves!" How many knights were there among them?

9.23. Three inhabitants of the island were asked, "How many knaves are there among you?" The first answered "none", the second answered "one". Who is the third inhabitant, a knight or a knave?

9.24. Three inhabitants of the island were asked how many knights are there among the other two. The first answered "none". The second answered "one". What did the third say?

9.25. In a room there are 20 chairs of two colours, red and blue. Sitting on each of the chairs there is a knight or a knave. Each of them claimed that he is sitting on a blue chair. After they had changed their seats in some way, half of them said they are now sitting on blue chairs, the other half, on red. How many knights are now sitting on red chairs?

9.26. Aladdin is standing near the entrances to two caves. He knows that there is a treasure in one of the caves; on the other hand, in another cave a crocodile lives that swallows everyone who dares enter the cave.

Both caves are looked after by one guard, and Aladdin is allowed to ask him one question. Moreover, Aladdin knows that the guard tells the truth or lies alternatively from day to day.

What question could Aladdin ask in order to determine in which cave the treasure is?

9.27. Johnny and Freddy are twins. One of them is a knight, the other, a knave. Propose how to find out the names of the brothers by means of one yes-no question addressed to one of them.

9.28. Peter was a guest of the twin brothers Johnny and Freddy, and he knew that one of them never tells the truth. He asked one of them, “Are you Johnny?”, got the answer “Yes”, then addressed the same question to the other, got an answer as well, and this allowed Peter to determine who is who. So which of them is Johnny?

9.29. Some of the inhabitants of the Knight-Knave Island declared that among the population of the island, the number of knights is even. Others declared that among the population of the island, the number of knaves is odd. Can the number of inhabitants of the island be odd?

9.30. There are 12 knights and knaves in a room (there is at least one knight and at least one knave). The first said, “There are no knights here,” the second, “There is no more than 1 knight here,” the third, “There is no more than 2 knights here,” ..., the twelfth, “There is no more than 11 knights here.” How many knights were there in the room?

9.31. A survey was conducted among 100 knights and knaves: “Not counting you, are there more knights than knaves among you?”. When 51 persons had answered, and all of them said that there were more knaves than knights, the survey was suspended. How many knights were there?

Miscellaneous problems

9.32. A son of a professor is talking to the father of a son of this professor, and that professor is not engaged in this conversation. Is that possible?

9.33. There are 85 red and blue balloons in a room. It is known that

(1) at least one of the balloons is red;

(2) in any pair of balloons, at least one is blue.

How many red balloons are there in the room?

9.34. Three persons are speaking: Whitey, Blacky, and Reddy. The brunet said to Whitey, “It is amusing that one of us is a brunet, another is blond, and the third is a redhead, but for none of us the colour of hair corresponds to our names.” What is the colour of hair of each of the speakers?

9.35. Four girls, Assya, Katya, Galya, and Nina, are standing in a circle and speaking. The girl in a green dress (neither Asya nor Katya) is standing between the girl in a blue dress and Nina. The girl in a white dress is standing between the girl in a pink dress and Katya. What is the colour of the dress of each of the girls?

9.36. When three girls, Nadya, Valya, and Masha, went for a walk, they were wearing red, blue and green dresses. Their shoes were of the same colours, but only Nadya's shoes were of the same colour as her dress. Besides, neither Valya's shoes nor her dress were blue and Masha's shoes were red. Determine the colour of the dresses and shoes of each of the girls.

9.37. Of Johnny's pets, all the animals, except two, are dogs, all, except two, are cats, all, except two, are parrots and all the others are cockroaches. How many animals does he have and what are they?

9.38. The natural numbers 1 to 5 are written on five cards. Lyosha and Dima took two cards each without looking at them, and hid the remaining card without looking at it. Having looked at his cards, Lyosha told Dima: "I know that the sum of the numbers on your cards is even" and was correct. What numbers were written on Lyosha's cards?

9.39. In the assertion in quotation marks below, replace the dots by digits so that it will be true: "This sentence contains the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; the digit 0 occurs ... time(s), 1 occurs ... time(s), 2 occurs ... time(s), 3 occurs ... time(s), 4 occurs ... time(s), 5 occurs ... time(s), 6 occurs ... time(s), 7 occurs ... time(s), 8 occurs ... time(s), 9 occurs ... time(s)."

9.40. One day on the stairs I found a strange notebook. It contained the following 100 assertions: "In this notebook there is exactly 1 false assertion", "In this notebook there are exactly 2 false assertions", ..., "In this notebook there are exactly 100 false assertions". Which of these assertions are true?

9.41. Vika, Sonya, Borya, Dennis, and Alla are in line at the cafeteria. Vika is ahead of Sonya, but after Alla; Borya and Alla do not stand next to each other, Dennis is neither next to Alla, nor to Vika, nor to Borya. In what order are they standing?

9.42. Four paper figures lie in line on a table: a triangle, a disk, a rectangle, and a rhombus. They are coloured differently: red, blue, yellow, and green. It is known that the red figure lies between the blue and green ones; the rhombus lies to the right of the yellow figure; the disk lies to the right of both the triangle and the rhombus; the triangle is not at one of the extremities; the blue and yellow figures are not next to each other. Determine the order of the figures and the colour of each.

9.43. The fairytale strongmen Muromets, Nikitich, and Popovich were given 6 coins for their exploits: 3 gold ones and 3 silver ones. Each got

2 coins. Muromets does not know what coins were given to Nikitich, nor what coins were given to Popovich, but he does know, of course, what he was given. Invent a question for Muromets with possible answers "yes", "no", and "I don't know" such that the honest answer to it will allow you to determine what coins Muromets got.

9.44. Four logicians A, B, C, and D sit at a round table clockwise in the order indicated. They were shown nine cards of the same suit (six, seven, ... , king, ace), then the cards were shuffled and one card was given to each so that they could only see the card they were given. The logicians were asked, in order, the same question: "Is your card higher than that of your neighbour to the right?", to which they all answered "I don't know." What card did D get?

Fair sharing

9.45. Petya and Vassya bought a pie. Petya cut the pie into two pieces that he considered equal and wanted to take one of them. Vassya did not agree. What should they do?

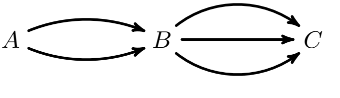
9.46. Petya, Kolya, and Vassya bought a pie. How should it be shared among them so that no conflicts arise?

9.47. Suggest a plan of fair sharing a pie between four boys.

Combinatorics

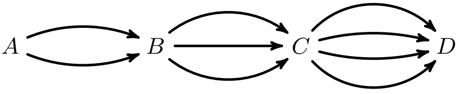
Combinatorics is the part of mathematics which deals with counting the number of possible variants (combinations) in different situations. For example, Problems 1.21–1.24 from Chapter 1 belong to combinatorics.

9.48. The cities A and B are joined by two roads, and the cities B and C, by three roads (see the figure). How many different routes joining A and C are there? The motion is one way, only in the direction of the arrows.

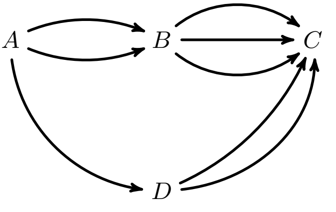


9.49. Two types of envelopes and three kinds of stamps are sold at the post office. In how many different ways can an envelope and a stamp be chosen?

9.50. To the roads from Problem 9.48, four roads from the city C to the city D were added (see the figure). How many routes from A to D are there now?



9.51. The cities A, B, C, and D are joined by one-way roads as shown in the figure. How many routes joining A and C are there?

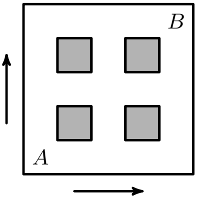


9.52. Ten hikers set out on a trip.

a) In how many ways can we choose two persons among them so that the first one will set up the tent and the second one, bring a pail of water?

b) In how many ways can we choose two persons among them so that they will bring two pails of water?

9.53. a) The figure shows a plan of 4 blocks and the adjoining streets in a city. How many routes joining point A to point B are there if one is allowed to move only in the directions shown by the arrows?



b) Solve the same problem for the case of $16 = 4 \cdot 4$ blocks rather than 4.

9.54. In how many ways can four different objects be placed in two different boxes, with two objects being placed in each box?

9.55. In how many ways can one put three identical coins in two pockets?

9.56. In how many ways can one put three different coins in two pockets?

9.57. How many different four-digit numbers divisible by 4 can be constructed from the digits 1, 2, 3, and 4 if:

a) each digit can occur only once;

b) each digit can occur several times or not occur at all?

9.58. A person has 10 friends whom he invites so that the composition of guests is always different (on one of the days he can invite nobody). How many days can he do that?

9.59. Among 49 schoolchildren each is acquainted with at least 25 others. Prove that they can be divided into groups of two or three persons so that each member of each group is acquainted with all the other members of this group.

9.60. During the year, 20 sessions of the San Seriffe parliament were held. Exactly 5 MP's were present at each session, and no two MP's met more than once at the same session. Prove that at least 21 MP's in all attended these sessions.

9.61. Among any 10 of 60 given students there are at least 3 classmates. Is it true that among these 60 students there must be at least

a) 15 classmates;

b) 16 classmates?

Inclusion and exclusion of properties

9.62. In a class, all the students are interested in mathematics or biology (or both). How many students are there in the class if 15 students are interested in mathematics, 21, in biology, and 10, both in mathematics and biology?

9.63. During the school break, 24 pupils bought ice cream. Fifteen of them bought chocolate ice cream, 17 bought strawberry ice cream. How many bought ice cream of both flavours?

9.64. Among 100 tourists, 5 people speak neither English nor German, 84 speak English, 56 speak German. How many tourists speak both these languages?

9.65. There were 2020 rose bushes in Anna's and Victor's garden. Victor watered half of the bushes, and Anna watered half of the bushes. It turned out that exactly 3 bushes — the most beautiful ones — were watered by both Victor and Anna How many bushes remained unwatered?

9.66. There are 70 children in a summer camp. Twenty seven of them participate in the drama studio, 32 sing in a choir, 22 are doing sports. In the drama studio, there are 10 children from the choir, 6 members of the choir are doing sports, and 8 of those participate in the drama studio. Three of those who are doing sports participate both in the drama studio and in the choir. How many children are not involved in anything of the above (i.e., they don't sing in the choir, nor do sports, nor participate in the drama studio)?

9.67. Each of the 4 sessions of a math circle was attended by 20 pupils. Nine pupils attended 3 of the 4 sessions, five pupils attended exactly 2, and three, only 1. How many pupils attended all the sessions?

9.68. There are bouquets of flowers in each room of a mansion. In all there are 30 bouquets of roses, 20 bouquets of carnations, 10 of chrysanthemums, and in each room there was at least one bouquet. In exactly two rooms there are both chrysanthemums and carnations, in exactly three rooms, there are both chrysanthemums and roses, in exactly four rooms, both carnations and roses. Could there be 54 rooms in this mansion?

Sets

9.69. In a herd consisting of horses, one-humped, and two-humped camels, there are 200 humps in all. How many animals are there in the herd if the number of horses equals the number of two-humped camels?

9.70. Twelve little children went out to the square to play in a sandpit. Each one who brought a toy bucket also brought a toy spade. Nine children left their buckets at home, two left their spades. Find the difference between the number of children that brought a shovel but left a pail at home and the number of children that brought a pail.

9.71. Pupils in a class were solving two problems. At the end of the lesson the teacher compiled four lists: list I for those who solved the first problem, list II for those who solved only one problem, list III for those who solved at least one problem, and list IV for those who solved both problems. Which of these lists will necessarily be longest? Can two of these lists be the same? If so, which ones?

9.72. Yura, Lyosha, and Misha collect stamps. The number of stamps in Yura's collection that Lyosha does not have is less than the number of stamps that both Yura and Lyosha have. Similarly, the number of Lyosha's stamps that Misha doesn't have is less than the number of stamps that both Lyosha and Misha have. Finally, the number of stamps that Misha has, but Yura has not, is less than the number of stamps that both Yura and Misha have. Prove that there is at least one stamp that all the boys have.

9.73. In the first pencil case there is a violet pen, a green pencil, and a red eraser; in the second, a blue pen, a green pencil, and a yellow eraser; in the third, a violet pen, an orange pencil, and a yellow eraser. The contents of the pencil cases is characterized by the following pattern: in any two of these cases only one pair of objects coincide both in colour and in what they are used for. What must be placed in the fourth case to follow this pattern? (Each of these cases contains one pencil, one pen, and one eraser.)

9.74. Petya has a toy locomotive with 12 cars, among which some are different, others are the same. Prove that from them Petya can compose as many 12-car trains as 11-car trains. (Trains are considered identical if the cars of the same kind appear at the same places.)

9.75. Each of the pupils in a class participates in no more than two school circles, and for any pair of pupils there is a circle in which they both participate. Prove that there is a circle in which no less than 2/3 of the whole class take part.

9.76. Three clowns, Bim, Bom, and Bam, came out on the arena of the circus in shirts of three colours: red, green, and blue. Their shoes were of the same three colours. Bim's shirt and shoes were of the same colour. Neither Bom's shoes nor his shirt were red. Bam wore green shoes and a shirt of a different colour. What were the clowns wearing?

9.77. A store acquired dresses of tree different styles and three different colours. A salesperson wants to choose three dresses for the showcase so that all the styles and all the colours are represented. Is this always possible?

9.78. There are several pitchers on a shelf, among them, at least two have different shapes and at least two are of different colours. Prove that there are two pitchers both of different shapes and of different colours on this shelf.

9.79. Among four persons, no three have the same first name, or the same middle name, or the same family name, but for any two either the first name, or the middle name, or the family name is the same. Is this possible?

9.80. During a conference, a mathematician figured out that among the mathematicians present every seventh is a philosopher, and among the philosophers present every ninth is a mathematician. Which is greater, the number of mathematicians or the number of the philosophers at the conference?

9.81. Prove that the quantity of five-digit numbers not divisible by 5 equals the quantity of five-digit numbers for which neither the first digit, nor the second digit from the left is the digit 5.

Chapter 10

How to Act

How many should one take?

10.1. In a dark room there is a vase containing several black and white balls. How many balls should one take from the vase to be sure that among them there are two balls of the same colour?

10.2. In a dark room there is a chest containing 10 black balls and 12 white balls. How many balls should one take from the chest to be sure that among them there are at least two black balls?

10.3. In a dark room there is a chest containing rubies, diamonds, and sapphires. What is the minimal number of jewels that one should take from the chest to be sure that among them there are at least two jewels of the same type?

10.4. In a dark room there is a chest containing 25 rubies, 15 diamonds, and 4 sapphires. What is the least number of jewels that one should take from the chest to be sure that that there are at least two diamonds among them?

10.5. A vase contains 20 balls, some of them are white, the rest are black. If one takes 10 balls from the vase, one of them will necessarily be white. If one takes 12, one of them will necessarily be black. How many black and how many white balls are there in the vase?

10.6. Petya has several coins in his pocket. If Petya takes any 3 coins from his pocket, there will necessarily be a one-ruble coin among them. If Petya takes any 4 coins from his pocket, then there will necessarily be a two-ruble coin among them. Petya took 5 coins. What were they?

Pouring liquids

10.7. Is it possible to distribute 50 litres of water in three tanks so that the first tank should contain 10 litres more water than the second one, and after pouring 26 litres from the second to the third tank, the third tank should contain as much water as the first one?
10.8. One has a 3 litre vessel and a 7 litre vessel. How can one, using these two vessels, pour into a pot 5 litres of water from a tap?
10.9. One has a 3 litre vessel and a 5 litre vessel. How can one, using these two vessels, pour 4 litres of water from a tap into the greater vessel?
10.10. One has a 4 litre vessel and a 9 litre vessel. How can one, using these two vessels, pour 6 litres of water from a tap into the greater vessel?
10.11. One has a 5 litre vessel and an 8 litre vessel. How can one, using these two vessels, pour 7 litres of water from a tap into the greater vessel?
10.12. One has a 5 litre vessel and a 7 litre vessel. How can one, using these two vessels, pour 6 litres of water from a tap into the greater vessel?
10.13. One has a 5 litre vessel and a 17 litre vessel. How can one, using these two vessels, pour 13 litres of water from a tap into the greater vessel?
10.14. One has a 15 litre vessel and a 16 litre vessel. How can one, using these two vessels, pour 8 litres of water from a tap into one of them?
10.15. There is a full twelve-litre can of milk and two empty 8 litre and 5 litre cans.

a) How can one divide the milk into two parts, 3 l and 9 l?

b) How can one divide the milk into two equal parts?

10.16. There is a full eight-litre can of milk and two empty 5 litre and 3 litre jugs. How can one divide the milk into two equal parts?
10.17. There is a full ten-litre can of milk and two empty 3 litre and 7 litre jugs. How can one divide the milk into two equal parts?
10.18. There are four barrels. The first contains 24 pails, the second, 13, the third, 11, the fourth, 5. The first barrel is filled with water, the other barrels are empty. How can one divide the water into three equal parts?
10.19. In a barrel there is no less than 13 pails of petrol. How can one separate out 8 pails of petrol using a nine-pail barrel and a five-pail barrel?
10.20. In a barrel there is no less than 10 pails of petrol. How can one separate out 6 pails of petrol using a nine-pail barrel and a five-pail barrel?

10.21. We have two full ten-litre cans of milk and two empty pots, of volumes 5 l and 4 l. Pour 2 litres of milk in each of the pots.

10.22. There remains 18 litres of petrol in a barrel. We have two seven-litre pails, and we are to pour 6 litres in each of the two pails. How can this be done if a four-litre scoop is available?

10.23. Tania is standing by the river. She has two clay jugs, a five-litre one and another about which she only knows that it is either a four-litre or a three-litre one. Help Tania find the capacity of the second one. (Looking into the jug, one cannot understand how much water it contains.)

Measuring out

10.24. There are two hourglasses, a 7-minute one and a 11-minute one. An egg must boil for 15 minutes. How can we measure out this period of time by means of the given hourglasses?

10.25. How is it possible to divide 24 kg of sand into two parts of 9 and 15 kg by means of balance scales without weights?

10.26. Each of two fuses burns non-uniformly, but burns out in precisely one hour. How can one, with these two fuses, measure out exactly 45 minutes?

Weighings

10.27. Three identical apples and 4 identical pears together weigh as much as 5 apples and 3 pears. What weighs more, one apple or one pear?

10.28. Three apples of the same size weigh as much as 4 pears of the same size. What is heavier, 4 apples or 5 pears?

10.29. We have balance scales and one weight of mass 100 grams. In three weighings, measure out 700 grams of rice.

10.30. We have balance scales and two weights of mass 50 g and 200 g.

a) In three weighings, measure out 2 kg of rice.

b) Can this be done if we only have one 200 g weight?

10.31. Propose a collection of four weights by means of which one can weigh any load whose mass is a whole number of grams, from 1 to 15, provided that the weights can be placed on only one of the plates.

10.32. A jeweller made six apparently similar silver ornaments weighing 22 g, 23 g, 24 g, 32 g, 34 g, and 36 g, and asked his apprentice to engrave the mass of each ornament on it. Can the apprentice check that he didn't confuse the ornaments by means of two weighings on balance scales without any weights?

10.33. On the table, there were 6 apples (not all of the same weight). Julia placed 3 of them on one of the plates of her scales and 3 on the other plate, and the scales turned out to be in equilibrium. Now Bill placed the same apples differently: 2 apples on one plate and 4 apples on the other, and the scales were in equilibrium again. Prove that one can place 1 apple on one of the plates and 2 on the other so that the scales will be in equilibrium.

Weighing coins

In the problems of this subsection, all the weighings are performed by means of balance scales without weights. Such a weighing only allows to compare the mass of the loads placed on the plates, i.e., to verify that they are equal or find out which of them is lighter.

10.34. Of 3 apparently similar coins, 1 is counterfeit, and it is lighter than the true ones. Find it in one weighing.

10.35. Of 9 apparently similar coins, 1 is counterfeit, and it is lighter than the true ones. Find it in two weighings.

10.36. Of 4 apparently similar coins, 1 is counterfeit, and it is lighter than the true ones. Find it in two weighings.

10.37. Of 8 apparently similar coins, one is counterfeit, and it is lighter than the true ones. Find it in two weighings.

10.38. Of 4 apparently similar coins, one is counterfeit, and its weight differs from that of the true ones (it can be lighter or heavier). Find out in two weighings which coin is counterfeit.

10.39. Of 3 apparently similar coins, 1 is counterfeit, and its weight differs from that of the true ones (it can be lighter or heavier). Find out in two weighings which coin is counterfeit and whether it is lighter or heavier than the true coins.

10.40. Soviet coins of denominations 1, 2, 3, 5 kopecks weighed 1, 2, 3, 5 g respectively. Among four such coins (one of each denomination) there is one counterfeit coin whose weight differs from that of the true one (it can be lighter or heavier). How can one find out in two weighings which coin is counterfeit?

10.41. There are 13 gold and 14 silver coins exactly one of which is counterfeit. It is known that if the counterfeit coin is gold, then it is lighter than the true gold coin, while if it is silver, then it is heavier than the true silver coin. How can one find the counterfeit coin in three weighings?

Crossing the river

10.42. Three hikers must cross a river. They have a boat that can only carry a weight of 100 kg. The first hiker weighs 45 kg., the second one, 50, the third, 80. How could they all cross the river?

10.43. Two boys were in a boat. The river shore was approached by a platoon of soldiers. The boat is so small that it can only carry one soldier or two boys. How can the soldiers cross the river?

10.44. A boatman is to take a wolf, a goat, and a cabbage across the river. In the boat he can only take either the wolf by itself, or the goat by itself, or the cabbage by itself. Besides, the cabbage cannot be left without supervision with the goat, and the goat cannot be left with the wolf. How can the crossing be carried out?

10.45. Three prison guards and three prisoners must cross the river. They have found a boat which can carry only two persons. The guards cannot stay on the shore if they are outnumbered by prisoners. How can all of them cross the river?

10.46. A family came to a bridge at night. The father can cross the bridge in 1 minute, the mother in 2, the son in 5, the grandmother in 10 minutes. They have only one flashlight. The bridge can only hold two persons. How can they cross the bridge in 17 minutes? (If two persons are crossing the bridge, they must walk at the speed of the slowest one. It is impossible to cross the bridge without the flashlight. To illuminate the bridge by the flashlight from far away is also impossible, nor is it possible to throw the flashlight over the river.)

10.47. Three knights and their three squires are to cross a river in a boat that can carry only two persons. The squires refuse to be together with knights other than their own if their master is not present. How could they cross?

10.48. Three missionaries and three cannibals must cross a river in a boat that can carry only two persons. The missionaries are afraid of being outnumbered. Only one missionary and one cannibal know how to row the boat. How could they cross?

10.49. The evil magician Koschey captured 43 persons and took them to his island. Prince Ivan left for the island in a two-place boat to rescue them. Meeting him, Koschey declared, "I'm sick and tired of feeding these parasites, let them get out of here on your boat. Have in mind that in order to reach the shore two persons are needed, but to get back, one will do. Before they begin, I will tell each of them about the 40 or more others that they are werewolves. You must decide yourself to whom I will tell who is a werewolf. If a prisoner has heard from someone that this person is a werewolf, then the prisoner will not get on the same boat with that person, but can stay on shore with that person. While there is at least one captive on the island, you are not allowed to take the boat; only when there is no one left can one of the former captives fetch you from the island. And if you do not succeed in organizing their return, you will stay with me forever." Does Ivan have the means to overcome this trial and return to the shore with the captives?

Order of steps

In all the problems of this section one is to find the sequence of steps that will lead to the required result. Problems related to pouring liquids, weighings, and river crossings are problems of this type.

10.50. Three hedgehogs are sharing three pieces of cheese of mass 5 g, 8 g, and 11 g. The fox offered her help. She can choose any two pieces and eat off 1 g from each of them. Can she, by repeating this procedure, leave three equal pieces of cheese to the hedgehogs?

10.51. There are five coins aligned on the table: the middle one, heads up, the others, tails up. One is allowed to turn over any three successive coins. Is it possible to get all five coins to show heads after several such operations?

10.52. There are six closed suitcases and six keys to them, but one doesn't know which key fits which suitcase. Is it true that one can always find this out in 15 attempts at most?

10.53. Twelve smiths must shoe 15 horses. It takes each smith 5 minutes to shoe one foot of any horse. What is the shortest possible time in which this can be done? (A horse cannot stand on two legs.)

10.54. Prince Ivan decided to attack the dragon Gorynych, who has three heads and three tails. "Here is a magic sword for you," said the witch Baba Yaga. "In one blow you can cut off either one head, or two heads, or one tail, or two tails. Remember: if you cut off a head, a new one grows back; if you cut off a tail, two new ones grow back; if you cut off two tails, one head grows back; if you cut off two heads nothing grows back."

a) Will Prince Ivan be able to cut off all of the dragon's heads and tails in 9 blows?

b) Will Prince Ivan be able to cut off all of the dragon's heads and tails in less than 9 blows?

10.55. Fifteen bananas and 20 oranges were growing on a magic tree. Simultaneously, one can pick either one or two fruits from the tree. If one picks one fruit, it regenerates: one fruit of the same type grows back; if one picks two fruits of the same type, an orange grows back, if one picks two different fruits, a banana grows back.

a) In what order should one pick the fruit so that only one piece of fruit will remain?

b) Can you then determine which fruit it will be?

c) Can one pick the fruit so that none should remain on the tree?

10.56. In the elevator of an 18-floor building there are only two buttons. If you push the first button, the elevator moves up 9 floors, while if you press the second button, it moves down 7 floors. (If the elevator must go to a floor higher than the 18th or lower than the 1st, it will not move.)

a) How can one get from the 1st floor to the 2nd on this elevator?

b) How can one get from the 2nd floor to the 1st on this elevator?

10.57. A fox and two bear cubs are sharing 100 sweets. The fox divides the sweets into three piles; who will get which pile is decided by drawing lots. If the cubs get a different number of candies, the fox is to equalize their piles by taking for herself the difference between the greater and the smaller pile. After that all of them eat the sweets that they were given.

a) Figure out how the fox can divide the sweets into three piles so that she will get precisely 80 sweets.

b) Can the fox work things out so that as the result she will eat precisely 65 sweets?

10.58. On the table, the 10-volume collection of Chekhov's works is divided into two stacks. A boy can take one or more books off the top of stack and put them on the top of other stack. How can he put all the books in one stack beginning with volume 1 (at the bottom), followed by volume two, and so on, in 19 such operations (or less)?

10.59. What is the shortest time needed to fry three pieces of bread from both sides if only two pieces fit on the pan, and one minute is needed to fry a piece of bread on one side?

10.60. A building contains several apparently identical rooms; these rooms are joined by corridors so that the rooms and the corridors form a circle, and in each room there is a chandelier and a switch. A spy managed to get into one of the rooms. How can he determine the number of rooms forming this circle if he can walk in the house and turn on and turn off the lights in the rooms? Initially the light may have been on or off in the rooms, but the spy does not know where.

10.61. Petya secretly picks a two-digit number, we are trying to guess what it is. To do that, we write two-digit numbers on the board, and next to each of them Petya writes a plus sign if it is the chosen one, a minus sign if either the first or the second of its digits is correct (but not both), and nothing if both digits are wrong. How can one guess Petya's number by writing 10 numbers at most?

10.62. Prince Ivan wants to get out of a round room with 6 doors, 5 of which are locked. In one attempt, he can check 3 successive doors, and if one of them is open, he gets out. After each attempt, the witch Baba-Yaga locks the door that was open and unlocks one of the neighbouring ones (which one, Ivan does not know). How can he get out of the room?

10.63. The dragon imprisoned 6 gnomes in a cave and said: "I have 7 caps in all colours of the rainbow. Tomorrow morning I will blindfold all of you and put a cap on each of you, hiding one of the caps. After that I will take off the blindfolds, so you will be able to see the colour of the caps, but you will not be allowed to speak to each other. Next, each gnome will tell me the colour of the hidden cap (other gnomes must not know what you will tell me). If at least 3 answers are correct, I will let you all go, if less, I'll eat you all for lunch". What strategy should the gnomes adopt so as to save their lives?

10.64. Dorothy has 5 coins, of which one is counterfeit. Only the Wizard can tell the counterfeit coin from a true one. Dorothy can choose 3 coins, give one of them to the Wizard and then ask him if there is a counterfeit one among the remaining two. Dorothy knows that the Wizard will tell the truth if he is given a true coin and it will lie if he is given a counterfeit one. How can Dorothy determine which coin among the 5 is counterfeit by asking no more than three questions?

10.65. The evil magician Koschey kidnapped three princesses. Prince Ivan set out to save them. When he reaches Koschey, the latter says, "Tomorrow you will see 5 bewitched young ladies. Of them 3 are princesses, the other 2 are my own daughters. You won't be able to distinguish them, whereas they will be able to. I will come up to each of them and ask her about each of them (herself included), „Is this a princess?“ When answering, they can lie or tell the truth, but each of them must say “yes” exactly two times. If after that you correctly guess who are the princesses, I will let you go free. And if you guess which of the princesses is the eldest, which is the middle one, and which is the youngest, then you can take them with you.” Ivan can give the princesses a note telling them how they should answer Koschey’s questions. Does Ivan have a strategy that guarantees him:

a) to return safely;

b) to return safely and take princesses with him?

10.66. An electrician was called upon to repair a garland with 4 lined up lightbulbs, one of which is blown. It takes 10 seconds to remove a lightbulb and 10 seconds to screw a lightbulb in. (All the rest is done by the electrician in no time.) What is the shortest time in which the electrician can certainly determine which lightbulb is blown if he has only one spare lightbulb?

Games

In this section, we are interested in games involving two players. Sometimes it happens that the result does not depend on the moves — the same player always wins. In other games, one of the players can always answer his opponent’s moves so as to win no matter what moves his opponent makes.

10.67. There are three heaps of stones: 10 stones in one, 15 in the second, 20 in the third. In one move the players are allowed to split any heap in two smaller ones; the player who cannot perform any move loses. Who wins, the first player or the second one?

10.68. There are 100 sweets in the first pile, 200 in the second pile. In one move the player must take any number of candies from one of the piles. The winner is the one who takes the last candy. Which of the players has a winning strategy?

10.69. Coming to school, Betsy and Suzy discovered an enigmatic inscription on the blackboard: “GORODCKAQ YCTHAQ OLIMPIADA”. They agreed on playing the following game: in one move each one can erase any number of identical letters, and the winner is the one who erases the last letter. Betsy made the first move and erased the last letter A. Find a winning strategy for Suzy.

10.70. The money forgers Johnny and Freddy have printed 20 counterfeit bills and are now writing in their seven-digit numbers. On each bill there are 7 slots for the number. The forgers act as follows: Freddy names either the digit 1 or the digit 2 (he doesn't know any others), then Johnny writes in that digit into any empty slot on any one of the bills and shows the result to Freddy, then Freddy names another digit (again 1 or 2), and so on. When all the slots are filled, Freddy takes as many bills with different numbers as he can (no two bills he has taken can have the same number), and the rest is taken by Johnny. What maximal number of bills can Freddy get, no matter how Johnny plays?

10.71. Two numbers 2014 and 2015 are written on the blackboard. Petya and Vassya are playing the following game. In one move, the player can either decrease one of the numbers by its nonzero digit or by a nonzero digit of the other number, or divide one of the numbers by 2 if it is even. The winner is the one who first writes a one-digit number. Petya and Vassya move in turn, Petya begins. Which of the players has a winning strategy?

10.72. Two pirates, Bill and John, each of whom has 74 gold coins, play the following game: in turn they put one, two, or three coins on the table, and the winner is the one who puts the one hundredth coin on the table. (Needless to say, each pirate can put only his coins on the table. If a player has no more coins for his move, he loses.) Bill begins. Which one of them has a winning strategy?

10.73. Having defeated the evil magician Koschey, Prince Ivan demanded some gold from him. Koschey took Ivan to the cave and said: "There are gold ingots in this chest, but you can't simply take them away, they are enchanted. You can take any number of them in your sack, but then I will take some of them from you (a different number from what you have taken) and put them back to the chest. So we will be putting the ingots back and forth, you to the sack, I to the chest, a different number of ingots each time. When further moves become impossible, you can take the sack with the ingots and leave". What is the maximal number of ingots that Prince Ivan can take with him if the chest contains

a) 13 ingots;

b) 14 ingots?

How should he do it?

Chapter 11

Extra Problems

Without repetitions

11.1. Fill in the cells of a 3 by 3 square table with the numbers 1, 2, and 3 so that in each row and each column each of these numbers should occur exactly once.

11.2. Fill in the cells of a 4 by 4 square table with the numbers 1, 2, 3, and 4 so that in each row, in each column, and in each of the diagonals each of these numbers should occur exactly once.

Equalities with matches

11.3. Erroneous equations with Roman numerals built of matches are presented below. For each of the equations (a)–(f), change the position of one match so as to make the equations true.

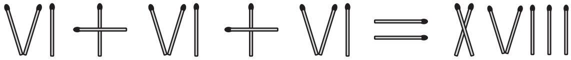
a) V = II + VIII; b) XII + IX = II;

c) X = VII − III; d) VI − VI = XI;

e) X + X = I; f) IV − V = I;

g) IV − I + V = II.

11.4. For each of the equations (a)–(c), change the position of one match so that the equations remain true.

a) 

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b) 

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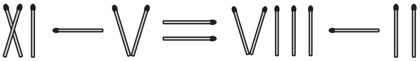
c) 

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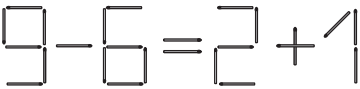
11.5. True equations with numbers built of matches are presented below. For each of the equations (a)–(c), change the position of one match so that the equations remain true.

a) 

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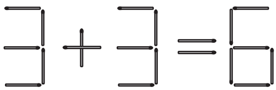
b) 

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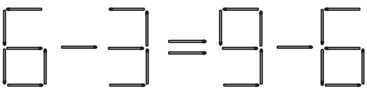
c) 

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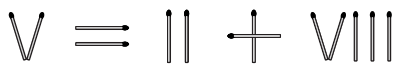
11.6. Two erroneous equations with Roman numerals built of matches are presented below. In each equation, change the position of one match so that the equation should become true.



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Invariants

In certain problems the following situation occurs. The given system successively changes its state and it is required to find out something about its final state. It may be difficult to trace all the changes, but sometimes the calculation of a certain quantity that does not change from state to state helps finding the answer to the given question (such a quantity is called an *invariant* of the system). Then it is clear that the value of the invariant in the final state is the same as in the initial one, i.e., the system cannot acquire a state with a different value of the invariant. The simplest and most frequently used invariant is the parity of a number; the invariant can be not only the remainder under division by 2, but also the remainder under division by some other number.

11.7. Several natural numbers are written on the blackboard with plus and minus signs between them. Is it possible to replace some of the pluses by minuses or vice versa so that the value of the expression should increase by 1?

11.8. A exchange machine exchanges any coin to five other coins. Johnny has one coin. Can he, using this machine, obtain:

a) 26 coins;

b) 27 coins?

11.9. Johnny and Freddy tore up a newspaper page in two. After that, they continued tearing the paper; if Johnny tore a piece of paper, he tore it into 7 pieces; if Freddy tore a piece of paper, he tore it into 13 pieces. In an attempt to restore the page, 99 fragments were found. Prove that some of the fragments were still missing.

11.10. The number -1 is written in one of the cells of an 8 by 8 square table; in all the other cells, 1 is written. You are allowed to simultaneously change the signs of all the numbers in any column or in any row. Prove that no matter how many times you do this, it will never happen that all the numbers are positive.

11.11. In an 8 by 8 square table one of the cells is black, all the other ones are white. Prove that by successive recolouring of columns or rows it is impossible to make all the squares white. By recolouring a column or a row, we mean changing the colours of all the cells in that column or row.

11.12. Starting from the a1 square on the chessboard, a knight returned to a1 after several moves. Prove that it made an even number of moves.

11.13. Starting from the a1 square, is it possible for a knight to pass through all the other squares exactly once and finish the route at h8?

11.14. A chess king passed through all the squares of the chessboard, visiting each square once, and returned to the initial square. Prove that it made an even number of diagonal moves.

11.15. The numbers $1, 2, 3, \ldots, 19, 20$ are written on the blackboard. You are allowed to erase any two numbers a and b and write the number $a + b - 1$ instead of them. What number can remain on the board after 19 such operations?

11.16. The numbers 0, 1, 0, 0 are written on the blackboard. One can choose any two of them and increase them simultaneously by 1. Is it possible to make all the numbers equal after several such operations?

11.17. There are seven glasses on the table, all of them upside down. It is allowed to turn over any four of them. Is it possible to make all the glasses stand right side up by means of several such operations?

11.18. m natural numbers are written on the blackboard in a row. One is allowed to add 1 to each n of these numbers ($n < m$). Is it always possible to make all the numbers equal by means of several such operations?

11.19. Ten ones and ten twos are written on the blackboard. Two players are playing the following game. In turn, they erase two numbers and, if they are equal, write the number 2 instead, while if they differ, the player writes number 1. If the last remaining number is 1, then the first player wins, if it is 2, the second one wins. Which player wins no matter how their opponent plays?

11.20. There are three heaps of stones. One is allowed to add to any one of them as many stones as there are in the two other heaps, or remove from any one of them as many stones as there are in the two others. For instance: $(12, 3, 5) \to (12, 20, 5)$ (or $(4, 3, 5)$). Is it possible, starting from the heaps of 1993, 199, and 19 stones, make one of the heaps empty?

11.21. There are six sparrows sitting on six trees, one bird on each tree. The trees form a straight line; the distance between any two neighbouring trees is 1 meter. If a sparrow flies from one tree to another, then some other also flies from one tree to another, the same distance, but in the opposite direction.

a) Can all the sparrows gather on one tree?

b) What if there are 7 sparrows and 7 trees?

11.22. Princess Vassilissa succeeded in locking up the evil magician Koschey in a straight corridor divided into four rooms by three passages. In each passage there is a fat tired guard leaning on a wall. Each time when Koschey moves from one room to another, the guard in the passage through which Koschey has passed, moves to the opposite wall and leans on it. If all the guards lean on the wall on the same side of the corridor, the wall will break and Koschey will escape. Can Vassilissa place Koschey and the guards so that Koschey will never escape?

The pigeonhole principle

Usually the *pigeonhole principle* is stated as follows: "If in n holes there are m pigeons and $m > n$, then there is a hole containing at least 2 pigeons". This obvious remark is often efficient when proving various statements.

11.23. Prove that in any football team there are two players who were born on the same day of the week.

11.24. Prove that among the inhabitants of Moscow there are at least 10 000 who celebrate their birthday on the same day. (It is known that there are more than five million inhabitants in the city.)

11.25. Twenty hikers set out on a trip. The oldest was 35, the youngest was 20. Is it true that among them, at least two must have the same age?

11.26. Is it possible to divide 44 marbles into 9 piles so that the number of marbles in all the piles should be different? (Every pile should contain at least one marble.)

11.27. There are 25 pupils in a class. It is known that for any two girls the number of boys in the class with whom they are friends is different. What is the maximal possible number of girls in that class?

11.28. Prove that among any twelve distinct two-digit natural numbers one can choose two numbers whose difference is a two-digit number with equal digits.

11.29. Prove that among the powers of 2 there are two numbers whose difference is divisible by 19.

11.30. Tidying the children's room for the arrival of guests, the mother found 9 socks. Among any 4 of these socks, at least 2 belonged to one child, while among any 5, no more than 3 belonged to a single child. To how many children did these socks belong and how many socks belonged to each of them?

11.31. Prove that in any group of people there are two persons who have the same number of friends in that group.

11.32. Prove that among any 6 persons there are either 3 persons who are acquainted with each other, or 3 pairwise unacquainted persons.

11.33. In a warehouse there are 200 boots of each of the sizes 41, 42, and 43, and among these 600 boots there are 300 for the right foot and 300 for the left foot. Prove that from them one can compose 100 pairs of a right and a left boot of the same size.

11.34. Given 25 coins of denomination 1, 2, 3, and 5 kopecks, will there necessarily be 7 coins of the same denomination among them?

11.35. In a mathematical olympiad 10 participants solved a total of 35 problems, and it is known that among them there are participants who have solved exactly 1 problem, participants who have solved exactly 2, and participants who have solved exactly 3 problems. Prove that there is a participant who has solved 5 problems.

11.36. Of four boys from the class 6“A” and four from 6“B” three were named Lyosha, three were named Vanya, and two were named Artyom. Could it happen that each one of them has a namesake in the same class?

11.37. The monkeys Masha, Dasha, Glasha, and Natasha ate a total of 16 bowls of porridge for lunch. Each monkey ate at least one bowl. Together, Glasha and Natasha ate 9 bowls, Masha ate more than Dasha, more than Glasha, and more than Natasha. How many bowls of porridge did Dasha eat?

11.38. Can 100 weights of masses $1, 2, 3, \ldots, 99, 100$ be divided into 10 piles of different masses so that the following condition holds: the heavier the pile, the less the number of weights is in it?

11.39. No matter how 30 pupils of a class are placed in a movie theatre, there will be at least 2 of them in some row. If there were 26 pupils in the class, then at least 3 rows would be empty. How many rows are there in the movie theatre?

11.40. Twenty birds flew into a photo atelier: 8 pigeons, 7 wagtails, and 5 woodpeckers. Each time when the photographer takes a snapshot, one of the birds flies out (forever). What is the maximal number of pictures the photographer can take in order to ensure that 4 birds of one type and 3 birds of another type will remain?

11.41. In a cat show 10 tomcats and 19 female cats are lined up, and next to each female cat there is tomcat that weighs more. Prove that next to each tomcat there is female cat that weighs less.

11.42. If one replaces the digits from 0 to 9 by the first ten letters from A to J (not necessarily in that order; different letters correspond to different digits), then the equation $\mathrm{AA} + \mathrm{A} = \mathrm{BCD}$ will hold (AA is the two-digit number both digits of which are A, BCD is the three-digit number with the digits B, C, and D, in that order). Find the last digit of the product $\mathrm{E} \cdot \mathrm{F} \cdot \mathrm{G} \cdot \mathrm{H} \cdot \mathrm{I} \cdot \mathrm{J}$.

11.43. Prove that among any 51 integers there are two for which the difference of their squares is divisible by 100.

11.44. Two magicians demonstrate the following trick. One person from the audience has 24 cards numbered from 1 to 24. He/she chooses 13 of them and hands them to the first magician, who returns two of them to the same person. The latter adds one card chosen from the 11 cards that he or she still has, shuffles these 3 cards and gives them to the second magician, who guesses which of the three cards was added by the person from the audience. How can this trick be performed?

Number systems

Ten digits are used in the decimal system, and they allow one to specify any natural number, however large. To do this, powers of 10 are used. One can similarly use powers of any other natural number. For instance, the notation using powers of 2 yields the *binary number system*. In that system there are only two "digits": 0 and 1. In order to distinguish binary notation from decimal notation, we add the subscript 2 to the number. In binary notation, $0_2 = 0 \cdot 1 = 1$, $1_2 = 1 \cdot 1 = 1$, $10_2 = 1 \cdot 2 + 0 \cdot 1 = 2$, $11_2 = 1 \cdot 2 + 1 \cdot 1 = 3$, etc. For instance,

$$1011_2 = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2 + 1 \cdot 1 = 8 + 0 + 2 + 1 = 11.$$

In the *ternary system*, three digits are used: 0, 1, 2. For the ternary system, we will use the subscript 3. For instance:

$$101_3 = 1 \cdot 3^2 + 0 \cdot 3 + 1 \cdot 1 = 10.$$

11.45. How is it possible to place 127 one-ruble coins into seven purses so as to be able to pay any sum from 1 to 127 rubles without opening any purse?

11.46. Misha chose an integer no smaller than 1 and no no greater than 1000. Vassya is allowed to ask only YES or NO questions (Misha always tells the truth). Can Vassya determine the number in 10 questions?

11.47. Write the number 61 in the ternary system.

11.48. One has balance scales and six weights of the masses 1, 2, 3, 9, 27, and 81 grams. How can one, using these weights, balance the scales on one plate of which a mass of 61 grams has been placed?

11.49. Using four weights of total mass 40 kg, Johnny wants to be able to weigh on balance scales any weight of integer mass from 1 to 40 kg. What weights does he need for that?

11.50. Prove that any natural number equals the difference of two numbers whose expression in ternary notation contains only the digits 0 and 1.

11.51. What is the least number of weights required to be able to weigh any integer number of grams from 1 to 100 on balance scales if we are allowed to place weights on both plates?

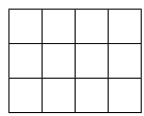
11.52. "You made a mistake," said Tania to Anton, sitting next to her, as she saw the equation $13^2 = 171$ in his notebook. "No," he answered, "it's just that I'm tired of the decimal system, and now I count in a system with a different base." "Which one?" asked Tania. Answer her question.

11.53. Prove that, replacing the stars in the expression $*1*2*4*8*16*32*64$ by pluses or minuses, one can obtain any odd integer number from -127 to 127, and in a unique way.

11.54. In the expression $*1*3*9*27*81$ you are allowed to delete any numbers with the stars preceding them and replace the remaining stars by pluses or minuses. Prove that one can obtain any nonzero integer from -121 to 121 in that way.

Tables

Tables consist of ordered *rows* and *columns*. For instance, the table shown here consists of 3 rows and 4 columns. One says that it is a 3×4-table. In the problems of this section, numbers are placed in the cells of tables.



11.55. Can one fill a 5×10-table with numbers so that the sum of the numbers in each row should equal 30, while that in each column, 10?

11.56. In each cell of an $m \times n$-table a number is written. The sum of numbers in each row as well as in each column equals 1. Prove that $m = n$.

11.57. The sixteen numbers: 1, 11, 21, 31, etc. are given (each number is obtained from the previous one by adding 10). Can one fill a 4×4-table with them so that the difference of any two numbers appearing in cells neighbouring side-by-side should not be divisible by 4?

11.58. The cells of a 5×5 table are filled with nonzero digits. The five digits in each row or in each column constitute a five-digit number. Can it happen that exactly one of these ten numbers is not divisible by 3?

11.59. Can an $n \times n$ table be filled by the numbers -1, 0, and 1 so that the sums in all the rows, in all the columns, and in the two diagonals differ?

11.60. Anna decided to fill a 5×8 table with digits so that each digit should occur in exactly 4 lines (here lines are either rows or columns). Prove that she will not succeed.

11.61. On the occasion of a holiday, 1 % of the soldiers in a regiment were given new uniform. The soldiers are lined up in a rectangle so that soldiers in new uniform occur in at least 30 % of the files and at least in 40 % of the ranks. What is the least possible number of soldiers in such a regiment?

Tournaments

To solve problems from this subsection, you need to know the following rules for organizing tournaments and counting points.

In football (association football) tournaments, 3 points are given for a victory, 1 point for a tie, 0 points for a loss.

In a chess tournament, participants are given 1 point for a win, half a point for a draw, zero points for a loss.

In volleyball tournaments there are no ties.

In a single round-robin tournament, each participant plays with each other participant once (in a double round-robin tournament, twice, and so on).

In a knockout tournament, the losing participant or team leaves the tournament.

11.62. Thirty teams played in a knockout tournament. How many games were played?

11.63. Can one organize a football tournament for 9 teams so that each team should play 3 matches?

11.64. Explain how one can organize a football tournament for 9 teams so that each team plays 4 matches.

11.65. In a single round-robin volleyball tournament, 20 % of all the teams won no matches. How many teams participated?

11.66. When a chess tournament ended, it turned out that each participant, playing with White, won the same number of games as all the other players together won playing with Black. Prove that all the participants had the same number of wins.

11.67. Sixteen teams participated in a single round-robin handball tournament (win = 2 points, tie = 1 point, loss = 0 points). All the teams earned a different number of points and the team that took seventh place earned 21 points. Prove that the winning team played to a tie at least once.

11.68. Three chess players A, B, and C played the same number of games with each other. Can it happen that A scored the greatest number of points, C scored the least, whereas A had the least number of wins and C, the greatest number of wins?

11.69. In a chess tournament, each participant played two games with each of the other participants, one with White, the other one with Black. All the participants scored the same number of points. Prove that there are two participants who won the same number of games with White.

11.70. Kostya, Misha, and Anton played ping-pong. After each game, the player who had lost it was replaced by the one who had not participated in it (e.g., if the first game was between Kostya and Misha and Misha won, then the second game was between Anton and Misha). Finally it turned out that Kostya played 8 games, while Misha played 17. Who lost the 5th game?

11.71. Several football teams are playing in a single round-robin tournament. Prove that at any moment in the tournament there are two teams that have played the same number of games.

11.72. Among 25 giraffes, all of different heights, a contest "Who is the tallest?" is conducted. In each round, five giraffes come out on the podium and the jury assigns them the numbers 1 to 5 in the descending order of height. How should the contest be organized so that after seven rounds the first, second, and third prizewinning giraffes could be found?

11.73. In a single round-robin chess tournament, Eugene and Alexander played the same number of games, but then fell ill and left the tournament. The remaining participants played out the tournament. Twenty three games in all were played in the tournament. Did Eugene and Alexander play against each other?

11.74. In a single round-robin chess tournament there were 8 participants and all scored a different number of points. The player who took 2nd place scored as many points as the players who took the last four places taken together. What was the result of the game between the players who took the 3rd and the 7th place?

11.75. In a single round-robin chess tournament there were 5 participants and all scored a different number of points. Only the player who took the second place lost no games; only the player who took the 5th place won no games. Determine the results of all the games.

11.76. In a single round-robin chess tournament there were 5 participants and all scored a different number of points. The player who took the first place drew no game; the player who took the second place lost no game; the player who took the 4th place won no game. Determine the results of all the games.

11.77. In a single round-robin chess tournament, among any 3 participants there is always one who scored 1.5 points in games with the other two. What is the maximal possible number of participants in such a tournament?

11.78. Several teams participated in a single round-robin football tournament. It turned out that the unique winner scored less than 50 % of the maximal number of points that one team could possibly score. What is the least number of teams that could have played in such a tournament?

11.79. Five teams participated in a single round-robin football tournament. Four of the teams scored 1, 2, 5, and 7 points. How many points did the 5th team obtain?

11.80. In a single round-robin football tournament, four teams participated. There was a team that scored the maximal number of points (each of the the other three teams scored less) and a team that scored the minimal number of points (each of the three other teams scored more). Is it possible that the former could have scored 2 points more than the latter?

11.81. In a single round-robin chess tournament there were 12 participants. According to the results of the tournament, the title of chessmaster was granted to any participant who earned more than 70 % of the maximal possible number of points. Can

a) 7 participants,

b) 8 participants

be granted the title of chessmaster?

11.82. There are 9 wrestlers of different strengths. In a bout between any two of them, the strongest always wins. Is it possible to divide them into 3 teams of 3 wrestlers each so that in the match ups "each against each", the first team defeats the second one (i.e., has more wins), the second defeats the third, and the third defeats the first?

11.83. Johnny and Freddy participated in a single round-robin chess tournament. Together they scored 6.5 points, while each of the other participants scored the same number of points. How many participants were there, besides Johnny and Freddy?

11.84. Twelve chess players participated in a single round-robin tournament. After that each wrote out 12 lists. The first list consisted of the player himself, ..., the list number $k + 1$ consisted those players that were in the kth list plus those whom the players of the kth list had defeated. It turned out that the 12th list of each of the players differed from their 11th one. How many games were drawn?

11.85. Prove that the participants in a single round-robin tournament can be numbered in such a way that no participant lost to the next one with respect to that numbering.

Computations

11.86. A very clever post office clerk acquired several packages with 100 envelopes in each. He counts out 10 envelopes in 10 seconds. In how many seconds can he count out 60 envelopes? 90 envelopes?

11.87. Forty children formed a circle and each held hands with both his/her neighbours. Among them, 22 held hands with a boy, while 30 held hands with a girl. How many girls were there in the circle?

11.88. Ten children stood in a row. Each of them gave an apple to the each girl or boy standing further to the right. After that the girls had 25 more apples than they initially had. How many girls were there in the row?

11.89. What is the value of the expression

$$(10^2 + 11^2 + 12^2 + 13^2 + 14^2) : 365?$$

11.90. Calculate

$$20202020^2 - 20202019 \cdot 20202021.$$

11.91. Prove that the number

$$2020^2 + 2020^2 \cdot 2021^2 + 2021^2$$

is the square of a natural number.

11.92. Prove that the number

$$2015 \cdot 2017 \cdot 2019 \cdot 2021 + 16$$

is the square of a natural number.

Probability

The *probability* of an event is the ratio of the number of favourable outcomes to the total number of all possible outcomes provided that all the outcomes are equally likely.

11.93. Two coins are tossed. If both come out heads, Petya wins, if one comes out heads, the other, tails, the winner is Vassya. Which one of them has higher probability of winning?

11.94. Two dice are thrown. If 11 points appear, Petya wins, if 12, the winner is Vassya. Which one of them has higher probability of winning?

11.95. The absent-minded Freddy is the first person entering the plane, and he occupies a random seat. Then the next passenger enters. If their seat is free, he or she occupies it, if not, they take a random seat. Then the next passenger enters and behaves in the same way, and so on. The total number of seats in the plane is 100, and they all are booked. What is the probability that the last passenger will occupy their seat?

11.96. A box contains 4 balls, each of which is either black or white. We are to guess how many balls of each colour there are in the box. At each step we are allowed to randomly choose and take two balls without looking into the box, look at them and put them back, after which the balls are mixed. Suppose 100 such experiments were performed, and in exactly 50 of them two black balls were taken. What do you think, how many balls of each colour were more likely to be in the box and why?

Miscellaneous problems

11.97. The digits of a 3-digit number A are written in reverse order; denote the resulting number by B. Is it possible that all the digits of the number $A + B$ are odd?

11.98. In the equation $101 - 102 = 1$ move one digit so that it should become true.

11.99. When "the day after tomorrow" becomes "yesterday", the distance from "today" to a Sunday will be the same as the distance from a Sunday to the day which was "today" when "yesterday" was "tomorrow". What day of the week is today?

11.100. In a certain town each family lives in their own house. Once each family moved to a house previously occupied by another family. On that occasion it was decided to paint each house in red, blue, or green so that for each family the colours of the old and new houses differ. Is this possible?

11.101. Vassya lives in an apartment house. At each entrance there is the same number of floors and four apartments on each floor, and each apartment has a 1-digit, or a 2-digit, or a 3-digit number. Vassya observed that the number of apartments with a 2-digit number in his entrance is 10 times the number of entrances in the house. How many apartments can there be in this house?

11.102. Thirty three strong men volunteer to guard the coast of Lukomoriye under the command of the clever Chernomor for 240 coins. Chernomor can divide the men into any number of teams (or unite them all into one team), and then he must distribute all the salaries between the teams. Then each team divides the coins equally among its men and returns the remaining coins to Chernomor. How many coins will be returned to Chernomor if

a) he distributes the salaries any way he wishes;

b) he distributes the salaries equally between the teams?

Answers

Chapter 1. Natural Numbers

1.1. 25. **1.2.** 35. **1.3.** 45. **1.4.** 27. **1.5.** 99 999. **1.6.** 962.
1.7. 1210 or 2020. **1.8.** 33. **1.9.** 41 312 432 or 23 421 314.
1.10. 2 100 010 006. **1.11.** 6 210 001 000.
1.12. 7 101 001 000 and 6 300 000 100.**1.13.** 68.
1.14. a) 97 531; b) 20 468. **1.15.** A girl. **1.16.** 7 girl classmates.
1.17. 5 cubes. **1.18.** The same number. **1.19.** There will be enough.
1.20. 8 flags. **1.21.** 6 numbers. **1.22.** 4 numbers. **1.23.** 18 numbers.
1.24. 24 numbers. **1.25.** 503 pills. **1.26.** 4 balloons.
1.27. 4 brothers and 3 sisters.
1.28. In the left pocket, 6 sweets, in the right one, 21. **1.29.** 6 siblings.
1.31. 6 boys and 4 donkeys. **1.32.** Tuesday. **1.33.** 18 seconds.
1.34. Yes. **1.35.** 178, 27, 57, 590, 2345, 36, 467, 14.

Chapter 2. Operations with Natural Numbers

2.1. $20 + 2000 = 2020$. **2.2.** The numbers are equal. **2.3.** 10 times.
2.4. 14. **2.5.** $175 : 5 = 35$. **2.6.** $100\,000 = 2^5 \cdot 5^5 = 32 \cdot 3125$.
2.7. 19, 25, and 6. **2.9.** a) No. b) No. **2.10.** 2. **2.11.** 8 pikes.
2.12. 3 kg. **2.13.** One third, or 8 hours.
2.14. One third. **2.15.** 2.4 rubles.
2.16. 8, 32, and 40 pencils in the first, second, and third box respectively.
2.17. In the first, 30 quarts, in the second, 50 quarts. **2.18.** 16 slices.
2.19. Nine tenths. **2.20.** 6 pencils.
2.21. 700 coins. **2.22.** There are 2 more silverfish than goldfish.
2.23. $1 + 2 + 34 + 56 + 7 = 100$ or $1 + 23 + 4 + 5 + 67 = 100$.
2.24. $9+8+7+65+4+3+2+1 = 99$ or $9+8+7+6+5+43+21 = 99$.

2.25. For instance, $2 \cdot 2 - 2 : 2 = 5 - 5 : 5 - 5 : 5$, or $22 : 22 = 55 : 5 - 5 - 5$, or $2 : 2 + 2 + 2 = 5 + 5 - 5 + 5 - 5$.

2.26. 7784, 854, 791, 161, 98, 35.

2.27. $1 - 2 + 4 + 8 - 16 - 32 + 64 = 27$. **2.28.** $1 \cdot (2 + 3) \cdot 4 \cdot 5 = 100$.

2.29. a) $(7 \cdot 9 + 12) : 3 - 2 = 23$; b) $(7 \cdot 9 + 12) : (3 - 2) = 75$.

2.30. a) $33 + 3 + 3 : 3$; b) $333 : (3 \cdot 3)$.

2.31. For instance, $(2 \cdot 2 \cdot 2 + 2) \cdot (2 \cdot 2 \cdot 2 + 2)$.

2.32.
$$\begin{aligned} 7 &= ((((1 : 2) : 3) : 4) : 5) : ((((6 : 7) : 8) : 9) : 10) = \\ &= 1 : ((2 : (3 : (((4 : 5) : 6) : 7))) : ((8 : 9) : 10)) = \\ &= 1 : ((2 : 3) : ((4 : ((5 : 6) : (7 : 8))) : (9 : 10))). \end{aligned}$$

2.33. 5050. **2.34.** 1222.

2.35. 100 g and 1800 g, 200 g and 1700 g, ..., 900 g and 1000 g, 1900 g.

2.36. 173, 174, 175, 176, 177. **2.37.** 1, 2, 3, 4, 5, 7. **2.38.** 45.

2.39. 240. **2.40.** 4995, independently of the order. **2.41.** $\begin{array}{r} 495 \\ +\;459 \\ \hline 954 \end{array}$.

2.42. $\begin{array}{r} 8126 \\ +\;8126 \\ \hline 16252 \end{array}$. **2.43.** $\begin{array}{r} 6823 \\ +\;6823 \\ \hline 13646 \end{array}$. **2.44.** $\begin{array}{r} 85679 \\ +\;85679 \\ \hline 171358 \end{array}$.

2.45. $2178 \times 4 = 8712$. **2.46.** $142\,857 \times 5 = 714\,285$.

2.47. $6 + 67 + 674 = 747$.

2.48. $28\,375 \times 3 = 85\,125$.

2.49. $8736 \times 18 = 157\,248$. **2.50.** $87\,172 \times 5 = 435\,860$.

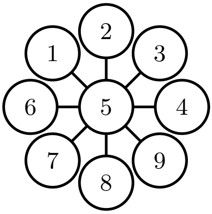
2.52. 2003. **2.53.** 6538.

2.54. 9991 and 1999. **2.55.** $\begin{array}{r} 305 \\ \times\;41 \\ \hline 305 \\ +\;1220 \\ \hline 12505 \end{array}$ or $\begin{array}{r} 315 \\ \times\;41 \\ \hline 315 \\ +\;1260 \\ \hline 12915 \end{array}$.

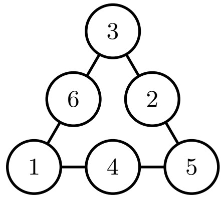
2.56. $\begin{array}{r} 7243 \\ \times\;29 \\ \hline 65187 \\ +\;14486 \\ \hline 210047 \end{array}$. **2.57.** $\begin{array}{r} 90809 \\ 12\,\overline{)1089708} \\ \underline{108} \\ 97 \\ \underline{96} \\ 108 \\ \underline{108} \\ 0 \end{array}$

2.58. $3128 : 23 = 136$.

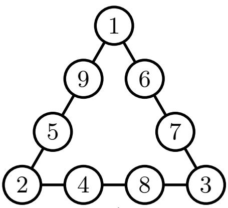
2.59. For instance, see the figure.



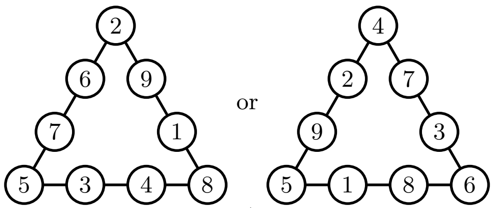
2.60. For instance, see the figure.



2.61. a) For instance, see Figure *a*); b) For instance, see Figure *b*).

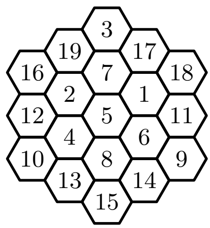


a)

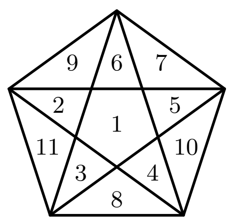


b)

2.62. $5, 8, 7, 5, 8, 7, 5, 8$. **2.63.** See the figure.



2.64. 1 1 2 1 2 2 3 3 1 3. **2.65.** See the figure.



2.66. 1001. **2.67.** 7. **2.68.** They can. **2.71.** $5^7 = 78\,125$.

2.72. $2^7 = 128$. **2.73.** $87^2 = 75\,69$. **2.74.** 15.

2.75.

5	9	1
7	2	6
3	4	8

.

2.76.

2	9	4
7	5	3
6	1	8

.

2.77. It's impossible.

2.78.

1	8	3
6	4	2
5	0	7

.

2.79.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

.

2.80.

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

.

Chapter 3. Word Problems

3.1. 11, 29, 21, or 19 leagues. **3.2.** 2 hr 30 min.
3.3. 10 minutes later. **3.4.** In 10 s. **3.5.** In 8 days.
3.6. In 15 min. **3.7.** 5 hours. **3.8.** One fourth of the route. **3.9.** 72 m.
3.10. In 3 days. **3.11.** 14 km.
3.12. It is impossible to gain that much time. **3.13.** 13 vessels.
3.14. 60 km/h. **3.15.** Grunty will win. **3.16.** By 40 min.
3.17. Yes. **3.18.** 17 min 15 s. **3.19.** 40 km/h.
3.20. Dmitry lives on the sixth floor. **3.21.** 15 years old.
3.22. 14 years old. **3.23.** 23 years old.
3.24. The son is 12, the father is 36.
3.25. The granddaughter is 5 years old, the grandmother is 60.
3.26. Mike is older. **3.27.** 12 years old. **3.28.** 5 cats. **3.29.** 2 min.
3.30. 20 beavers. **3.31.** Two logs. **3.32.** In 35 days. **3.33.** 400 g.
3.34. Joint work is cheaper. **3.35.** 200 rubles.
3.36. The small bird costs 10 rubles, the big one, 20.
3.37. 13 rubles. **3.38.** 38 rubles. **3.39.** 50 roubles. **3.40.** 2.50 rubles.
3.41. By 100 rubles. **3.42.** 7 soldo. **3.43.** 5 pups and 12 ducklings.
3.44. 4 cats and 6 dogs. **3.45.** 6 cars.
3.46. Three stools and four chairs. **3.47.** 20 mushrooms more.
3.48. 18 mushrooms. **3.49.** 6 children.
3.50. All the money should be given to the second one.
3.51. 3 rubles to the first and 12 rubles to the second.
3.52. 2 mackerels. **3.53.** The brown ones.
3.54. There are 22 more silver coins.
3.55. There are more white animals that are not cats. **3.57.** 30 kg.
3.58. All together 180 kg, Ken 40 kg, mother 60 kg, father 80 kg.
3.59. 1 pound. **3.60.** 6 times. **3.61.** Basil is right.
3.62. 13 questions. **3.63.** 11 blocks of wood. **3.64.** 20 min.
3.65. 6 logs. **3.66.** 120 cuts. **3.67.** 3 sheets. **3.68.** 22 squares.
3.69. The answer to both questions is 11 breaks.
3.70. 8 four-metre logs and 2 five-metre logs.

3.71. 21 pieces. **3.72.** 50 irons. **3.73.** 140 rubles.
3.75. 10 dollars. **3.76.** On the first channel. **3.77.** 23 persons.
3.78. Greg finished first, then, Alex, while Ellen was last.
3.79. By 6:03. **3.80.** 500 pages. **3.81.** At 4.
3.82. 21 raspberry muffins, 7 blueberry muffins, 14 strawberry muffins.
3.83. 20 yellow and 15 white dandelions.
3.84. a) 25 dandelions; b) 9 dandelions.
3.85. Any one, except Tigger. **3.86.** 30 on the birch, 5 on the alder.
3.87. 156 nuts. **3.88.** 15 bulbs.

Chapter 4. Divisibility of Natural Numbers

4.1. They can't. **4.2.** Impossible. **4.3.** No.
4.6. He did. **4.7.** Five girls. **4.8.** Odd. **4.9.** No.
4.10. No. **4.12.** Impossible. **4.14.** No.
4.15. No. **4.17.** No. **4.18.** He did make a mistake.
4.20. The first one wins regardless of the strategy. **4.23.** 33 numbers.
4.24. 5 zeros. **4.25.** Yes. **4.27.** 111 grains.
4.29. 59 040, 59 940, 59 544, and 59 148 **4.30.** 4104. **4.31.** 2 347 200.
4.32. 72 630 or 72 135. **4.33.** 92. **4.35.** 39 916 800. **4.36.** No.
4.37. 3 peanuts. **4.38.** No. **4.39.** No. **4.40.** Yes.
4.41. Yes. **4.42.** Two muffins. **4.43.** One.
4.44. They did make a mistake. **4.45.** Impossible.
4.51. Two copies of 27 or of 2710 027, or of 271 002 710 027, etc.
4.52. 8899. **4.53.** For instance, 13 029, 14 039, or 24 038.
4.54. For instance, 6, 10 or 15. **4.55.** 148. **4.58.** 4.
4.59. He is mistaken. **4.60.** For instance, $N = 12$. **4.61.** The last one.
4.63. Impossible. **4.65.** 9870. **4.66.** Nicholas. **4.67.** Yes for both.
4.69. For example, 1, 2, 4, 8, 975 360. **4.70.** Yes. **4.71.** a) Yes. b) No.
4.72. a) Yes, to the first question. b) No, not knowing the number, it is impossible.
4.73. 532 and 14 or 215 and 43.
4.74. He is right. **4.75.** 11 wheels of cheese. **4.76.** Yes.
4.77. 41. **4.78.** 8, 16, 24, 32, 40, 48. **4.79.** On Friday at 4 pm.
4.80. 10, 25, and 40. **4.81.** On the fourth floor.
4.82. a)–c) Yes. d) No. **4.84.** Yes. **4.85.** 60.
4.86. Flour in the 31 kg sack, salt in the 36 kg sack, sand in all the other sacks.
4.87. It cannot. **4.90.** 2.

4.93. 11, 22, 33, 44, 55, 66, 77, 88, 99, 12, 24, 36, 48, and 15.
4.94. They can't. **4.95.** The replaced digit is 6. **4.96.** Not always.
4.99. No. **4.100.** b) No. **4.101.** 763.
4.102. The index finger. **4.103.** 5. **4.104.** 205 numbers.
4.107. 329, 392, 518, 581.
4.108. For instance, B = 9, A = 1, and O = 0.
4.109. No. **4.110.** Yes to both questions.
4.111. a) Yes. b) Not necessarily. **4.112.** Squares of prime numbers.
4.113. 5 rubles. **4.115.** $p = 2$. **4.116.** 2 and 5.
4.117. a) 5; b) 2; c) only one.
4.118. The number 1111 with prime divisors 11 and 101.
4.119. 985 flats. **4.120.** 23. **4.121.** 30 or 24.
4.122. Impossible. **4.123.** 133. **4.124.** $p = 3$. **4.126.** $p = 3$.
4.127. $p = 5$. **4.130.** No. **4.131.** 2, 3, 5, 7, and 11. **4.132.** 5 years.
4.133. 2, 5 or 11. **4.134.** a) 4; b) 6; c) 9; d) $(m + 1)(n + 1)$.
4.135. 2009 or 1109.
4.136. a) For instance, $2312, \ldots, 2321$; b) $30, \ldots, 39$;
c) $22, \ldots, 31$; d) $3, \ldots, 12$; f) At most five.
4.138. 19.
4.139. $203 = 29 \cdot 7 \cdot 1 \cdot \ldots \cdot 1 = 29 + 7 + 1 + \ldots + 1$ (ones occur 167 times in each case).
4.140. 19 pupils contributed 2615 rubles each. **4.141.** $7, 5, 4, 3$, and 1.
4.142. $143 \cdot 14 \cdot 1 = 2002$. **4.143.** $29 \cdot 69 = 69 \cdot 29 = 2001$.
4.144. 7 floors. **4.145.** 7 watchmen. **4.146.** 72. **4.147.** 3.
4.148. 251. **4.149.** a) It's possible; b) it's possible.
4.150. 5 boxes of "Sweet Mathematics" and 4 boxes of "Geometry with Nuts".
4.151. May 30. **4.152.** For instance, 71 and 72. **4.153.** One paper.
4.154. 111. **4.155.** 839. **4.156.** 503.
4.157. LCM = 121 212 121 212, GCD = 121 212.
4.160. 3. **4.161.** 9. **4.162.** 24. **4.163.** 666, 1322, 1998.
4.164. 36 and 252; 108 and 180. **4.165.** 90 and 126.
4.166. They exist. **4.168.** Impossible.

Chapter 5. Fractions

5.1. 4 times. **5.2.** 19 m. **5.3.** 7 litres. **5.4.** 1 dollar 11 cents.
5.5. 21 children. **5.6.** $\frac{5}{4}$ min. **5.7.** 22.

5.8. a) $\frac{1}{3}$; b) $\frac{1}{5}$; c) $\frac{1}{7}$. **5.9.** $\frac{7}{15}$ and $\frac{8}{15}$. **5.10.** $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, and $\frac{7}{8}$.
5.11. $\frac{60}{72}$. **5.12.** $\frac{78}{79} < \frac{90}{91}$. **5.13.** $\frac{2}{5}$. **5.14.** 437. **5.15.** Yes.
5.17. Sophie is right. **5.18.** $\frac{2}{3}$ of the route.
5.19. Impossible. **5.20.** He did.
5.21. In $12\frac{4}{7}$ s. **5.22.** 135 m. **5.24.** 1 kg and 2 kg.
5.25. 18 l. **5.26.** It's possible.
5.28. 56 teabags. **5.29.** 15 holes. **5.30.** 10 holes.
5.31. $\frac{6}{7}$ and $\frac{1}{7}$. **5.32.** 28 hours. **5.33.** 108 km. **5.34.** $\frac{1}{n+1}$.
5.35. 6000 rubles. **5.36.** 6 trains. **5.37.** $\frac{1}{64}$. **5.38.** 8 mowers.
5.39. $\frac{3}{10}$. **5.41.** Between pages 430 and 431.
5.42. 365. **5.43.** a) 2020 fractions; b) 32 more. **5.44.** $\frac{2}{5}$.
5.45. a) For instance, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$; b) For instance, $\frac{2}{11}$, $\frac{3}{11}$, $\frac{6}{11}$.
5.46. It's possible. **5.47.** $\frac{25}{76}$. **5.48.** $\frac{1}{204}$.
5.49. For instance, 2, 3, $\frac{3}{2}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$.
5.50. Yes, Zhuchka did visit the second patch.
5.51. $\left(\frac{1}{2} - \frac{1}{6}\right) : \frac{1}{6060} = 2020$. **5.52.** $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$.
5.53. $9 + \frac{99}{9}$. **5.54.** $44 + \frac{44}{4}$. **5.55.** $99 + \frac{99}{99}$.
5.56. For instance, $\frac{1}{6} + \frac{7}{3} = \frac{5}{2}$. **5.57.** $\frac{20}{10} - \frac{10}{15} = \frac{20}{15}$. **5.58.** 6 km/h.
5.59. 32 years old. **5.60.** 8 tests. **5.61.** It will decrease by 6.
5.63. 70 points. **5.64.** 40 years old. **5.65.** It will decrease by 6.
5.66. Yes, at least one of them did. **5.67.** 10 decorators.
5.68. 80 windows. **5.69.** 16, 24, 48.
5.70. $a = 45$, $b = 30$, $c = 18$. **5.71.** $\frac{2}{3}$.
5.72. By one fourth. **5.73.** By $\frac{2}{3}$. **5.74.** The father is 45, the son is 20.
5.75. 7 floors. **5.76.** At 22:30.
5.77. 18, 36, 54, 72, 90; and 9, 18, 27, 36, 45.
5.78. By 300 %. **5.79.** 25 % more. **5.80.** By 25 %. **5.81.** 60 %.
5.82. 20 %. **5.83.** 750 g. **5.84.** 50 kg. **5.85.** 500 kg.
5.86. $\frac{3}{5}$ of the jam. **5.87.** It is true. **5.88.** 700 coins.
5.89. 11 .
5.90. 25 rubles and 18 rubles or 18 rubles and 25 rubles.
5.92. 16. **5.93.** $0.125 < 0.13 < \frac{27}{200}$.
5.94. $0.7 < \frac{37}{50} < \frac{3}{4}$. **5.95.** $0.4 < \frac{3}{7} < \frac{6}{13}$.

5.96. Decimal point. **5.97.** 40 cents. **5.98.** It is possible.
5.99. The decimal points may be placed in different ways:

$$20.16 + 20.16 + 20.16 + 201.6 + 201.6 = 463.68 \quad \text{or}$$
$$2.016 + 2.016 + 2.016 + 20.16 + 20.16 = 46.368.$$

5.101. No, it cannot. **5.102.** It will decrease by 4 %.
5.103. In both stores milk will cost the same once again.
5.104. It increased by 35 %. **5.105.** He lost weight.
5.106. It will decrease by 25 %. **5.107.** 30 %. **5.108.** Basil.
5.109. The amounts of water will be equal.
5.110. The result will be 1 % less than the correct one.
5.111. After four washings. **5.112.** 11 participants. **5.113.** 4 days.
5.114. Yes. **5.115.** By 50 %. **5.116.** 300 coins.
5.117. $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$. **5.118.** $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$. **5.120.** $\frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$.
5.122. $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$. **5.123.** $1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}$. **5.124.** $\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$.
5.125. $\frac{1}{7} + \frac{1}{42}$, $\frac{1}{8} + \frac{1}{24}$, $\frac{1}{9} + \frac{1}{18}$, $\frac{1}{10} + \frac{1}{15}$.
5.126. $\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$, $\frac{7}{10} = \frac{1}{2} + \frac{1}{5}$, $\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$,
$\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$, $\frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}$, $\frac{2}{5} = \frac{1}{4} + \frac{1}{10} + \frac{1}{20}$.
5.127. In two ways. **5.128.** In four ways. **5.129.** In seven ways.
5.130. $\frac{1}{4} + \frac{1}{8}$ or $\frac{1}{3} + \frac{1}{24}$. **5.131.** No, it cannot.

Chapter 6. Integers and rational numbers

6.2. For $n = -9, -7, -5$, or -3. **6.5.** 4.
6.6. $(1 - 2) \cdot 3 + (4 + 5 \cdot 6 \cdot 7 + 8) \cdot 9 = 1995$. **6.9.** 6 farthings.
6.11. a) No; b) yes. **6.12.** $\frac{3}{8}$ and $\frac{5}{13}$ or $\frac{5}{8}$ and $\frac{8}{13}$. **6.13.** 11.
6.14. 0.(027). **6.15.** 0.(142857).
6.16. 0.(285714); 0.(428571); 0.(571428); 0.(714285); 0.(857142). The pattern is the following: if we write the periodic parts in a circle, they will be the same.
6.17. 0.0(714285). **6.18.** 0.(0588235294117647).
6.19. $0.0(3) = \frac{1}{30}$, $0.00(3) = \frac{1}{300}$. **6.21.** $\frac{13}{99}$, $\frac{238}{999}$.
6.22. a) $\frac{4}{27} = 0.(148)$; b) $\frac{243\,334}{999\,999} = 0.(243334)$; the proper fraction can be reduced to $\frac{2674}{10\,989}$.
6.25. $\frac{14}{99} = 0.(14)$. **6.26.** 3, −4, 3, −4, 3 (in this and similar problems, the answer is not unique). **6.27.** 1, 1, −3, 1, 1.

6.28. 2, 2, −7, 2, 2. **6.29.** 6, −7, 6, −7, 6, −7, 6.
6.30. 4, 4, −9, 4, 4, −9, 4. **6.31.** Impossible. **6.32.** Impossible.
6.33. a) Yes; b) no; c) no. **6.34.** Impossible.
6.35. a) Yes; b) no. **6.36.** 3, 3, −8, 3, 3, 3, −8, 3, 3.
6.37. 5, 5, −13, 5, 5, 5, −13, 5, 5, −13, 5, 5, 5, −13, 5, 5.

Chapter 7. Equations

7.1. 1. **7.2.** 18 employees. **7.3.** Yes. **7.4.** 6 days. **7.5.** 8 kg.
7.6. 3 branches and 4 toys. **7.7.** None. **7.8.** 27 toys.
7.9. 40 kg. **7.10.** 160 g. **7.11.** $\frac{3}{2}, -\frac{1}{2}, \frac{5}{2}, \frac{1}{2}, \frac{7}{2}$.
7.12. The King is 28 years old, the Queen is 21 years old. **7.13.** 40 km/h.
7.14. The speed of the car that started at A is 2.5 times greater.
7.15. 480 km. **7.16.** 4.5 hours. **7.17.** 55° or 125°.
7.18. 4 hours. **7.19.** 37 km. **7.20.** In 2 min. **7.21.** 100 km.
7.22. In 29 days. **7.23.** 1237 mice. **7.24.** 27 plums.
7.25. 4, 2, and 1. **7.26.** 255 lemons. **7.27.** 27, 18, and 12 plums.
7.28. 7 matches. **7.29.** $21. **7.30.** 24 coins. **7.31.** $x = 9$.
7.32. Any integer from 100 to 109. **7.33.** No. **7.34.** 18.
7.35. 36. **7.36.** The first caught 36 fish, the second, 34.
7.37. The first caught 36 fish, the second, 44. **7.38.** 10.
7.39. 16. **7.40.** 10. **7.41.** 144.
7.42. 21 candies. **7.43.** 5 persons. **7.44.** Besides 20, there exists a unique number satisfying these conditions.
7.45. 37 and 73. **7.46.** Each was hit exactly once.
7.47. January 16, 2010. **7.48.** He can.
7.49. 1 bull, 9 cows, and 90 calves. **7.51.** 400 musicians.
7.52. $(6, 13)$, $(16, 3)$, $(4, -9)$, $(-6, 1)$. **7.53.** 4×4 or 3×6.
7.54. 3×10 or 4×6. **7.55.** Only 6. **7.56.** 647. **7.57.** In 1980.
7.58. Lisa's brother is Kolya, Dasha's brother is Petya, Anya's brother is Tolya, and Katya's brother is Vassya.
7.59. It can take 25 seven-meter jumps and 5 five-meter jumps,
or 26 seven-meter jumps, 3 five-meter jumps and 1 three-meter jump,
or 27 seven-meter jumps, 1 five-meter jump, and 2 three-meter jumps.
7.60. Uniquely. **7.61.** $x = 10$, $y = 3$. **7.62.** $x = 1$, $y = 2$.
7.63. $(3, 5, 7)$ and all possible permutations of these numbers.
7.64. $n = 4$ or 8. **7.65.** It is possible. **7.67.** There are no solutions.
7.68. $(0, 0, 0)$. **7.69.** $(n, m) = (2, 8)$ or $(6, 28)$. **7.70.** 5 and 2.6.
7.71. 117, 156, and 195.

7.72. The first, 814 books, the second, 1026 books, the third, 150 books.
7.73. 8 children. **7.74.** $n = m = 1$ or $n = 2$, $m = 4$.

Chapter 8. Inequalities

8.1. The second number is greater. **8.2.** $\frac{11}{16}$. **8.3.** $\frac{9}{26}$.
8.5. The Wizard ate one cake, the Tin Man ate 3, the Scarecrow ate 11, and the Lion ate 13.
8.6. 2 or 4 helpers. **8.7.** Porthos, d'Artagnan, Athos, Aramis.
8.8. $d > a > b > c$. **8.9.** No. **8.10.** No. **8.12.** Yes, it could.
8.14. Yes, it could. **8.15.** To the third floor. **8.16.** 3^{200}. **8.17.** 17^{14}.
8.18. The first number is greater. **8.19.** 2 and 3, 2 and 5, 3 and 11.
8.20. 7:59pm. **8.21.** 9 pikes. **8.22.** 1 999 999 999.
8.23. $(a + d)(b + c)$. **8.24.** 763, 852, and 941.
8.25. 3, 415, 43, 74, 7, 8. **8.26.** 6 bunnies.
8.27. 5 guests. **8.28.** 80 people. **8.29.** 29 899 856.
8.30. 36 min. **8.31.** 3000 dinars. **8.32.** 45 monkeys.
8.33. 3 painters and 6 fitters (the remaining worker can be considered either painter, or fitter, or he/she can do nothing — the time needed will not change).
8.34. 553 451 234 512 345.
8.35. a) 00 000 123 450; b) 10 000 012 340; c) 99 999 785 960.
8.36. 7 317 192 329.
8.37. From left to right: the Scarecrow, Dorothy, the Lion, the Tin Man.
8.38. 0, 1, 2, 3, 4, 5, 6, 7, 8, 10 mushrooms.
8.39. Lusya Egorov and Yura Vorobiev, Olya Petrova and Andrey Egorov, Inna Krymova and Seriozha Petrov, Anya Vorobieva and Dima Krymov.
8.40. The shortest one among the tallest (if they are of different height.)
8.41. Friday. **8.42.** Yes. **8.43.** 13 weights.

Chapter 9. Logic, Combinatorics, Sets

9.1. There is sugar in the first sack, wheat in the second one, flour in the third one.
9.2. The green one. **9.4.** 11. **9.5.** Nikitich.
9.6. Harry. **9.7.** Igor. **9.8.** 2. **9.9.** 10 books or none.
9.10. To the city A. **9.11.** 5 ladybugs. **9.12.** 7 boys.
9.13. Gesha is honest, Roma is clever, Kesha is a liar.
9.14. a) All were hobbits; b) 5 goblins and 5 elves.

9.15. Parrot, Lion, Giraffe, Jackal. **9.16.** A tourist. **9.17.** A knight.
9.18. Any question to which both the person asking and the person answering know the correct answer. For instance: "Is it raining?".
9.19. "Do you always tell the truth?"
9.20. "Did I ask you anything today?"
9.21. Johnny is a knave, Freddy is a knight. **9.22.** 1. **9.23.** A knave.
9.24. "One". **9.25.** 5. **9.28.** The second one. **9.29.** No.
9.30. 6. **9.31.** 50. **9.32.** Possible. **9.33.** Only one.
9.34. Whitey is a redhead, Blacky is blond, Reddy is a brunet.
9.35. Galya is in green, Katya, in blue, Assya, in white, Nina, in pink.
9.36. Nadya is wearing blue shoes and a blue dress; Valya is wearing white shoes and a red dress; Masha's shoes are red and her dress is white.
9.37. Either one dog, one cat, and one parrot, or just two cockroaches.
9.38. 2 and 4.
9.39. Instead of ellipses, write the digits $2, 2, 8, 4, 3, 2, 2, 2, 3, 2$.
9.40. Only the 99th. **9.41.** Alla, Vika, Borya, Sonya, Dennis.
9.42. Yellow rectangle, green rhombus, red triangle, blue disk.
9.43. For instance, "Is it true that you were given more gold coins than Popovich?"
9.44. A ten. **9.48.** 6. **9.49.** 6. **9.50.** 24. **9.51.** 8.
9.52. a) 90; b) 45. **9.53.** a) 6; b) 70. **9.54.** 6. **9.55.** 4. **9.56.** 8.
9.57. a) 6; b) 64. **9.58.** 1024. **9.61.** a) Yes; b) no.
9.62. 26 students. **9.63.** 8 people. **9.64.** 45 people. **9.65.** 3 bushes.
9.66. 10 people. **9.67.** 10 people. **9.68.** No.
9.69. 200 animals. **9.70.** 4.
9.71. List III is the longest. Possible coincidences: I and IV; II and III; I, II, and III; I, III, IV; I, II, III, and IV.
9.73. Green pen, orange pencil, red eraser.
9.76. Bim was wearing a red shirt and red shoes, Bom was wearing a green shirt and blue shoes, Bam was wearing a blue shirt and green shoes.
9.77. No, not always. **9.79.** Yes, this is possible.
9.80. There are more philosophers than mathematicians.

Chapter 10. How to Act

10.1. 3 balls. **10.2.** 14 balls. **10.3.** 4 jewels. **10.4.** 31 jewels.
10.5. 11 white balls and 9 black balls.
10.6. Three one-ruble coins and two two-ruble coins.

10.7. Impossible. **10.27.** A pear is heavier.
10.28. 4 apples weigh more. **10.30.** b) Yes.
10.31. 1, 2, 4, and 8 g. **10.32.** Yes. **10.49.** Yes.
10.50. Yes. **10.51.** Possible. **10.52.** Yes. **10.53.** 25 min.
10.54. a) Yes; b) no. **10.55.** b) A banana; c) no.
10.57. a) 10, 10, and 80 sweets; b) no. **10.59.** 3 min.
10.65. Yes in both cases. **10.66.** 60 sec. **10.67.** The second one.
10.68. The first one. **10.70.** 2. **10.71.** Petya. **10.72.** John.
10.73. 13 in both cases.

Chapter 11. Extra Problems

11.1.

1	2	3
3	1	2
2	3	1

11.2.

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

11.3. a) X = II + VIII;

b) XII − IX = III;

c) X − VII = III;

d) VI + V = XI or V + VI = XI;

e) XI − X = I;

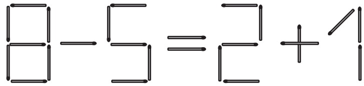
f) IV = V − I or VI − V = I;

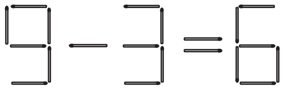
g) IV = I + V − II.

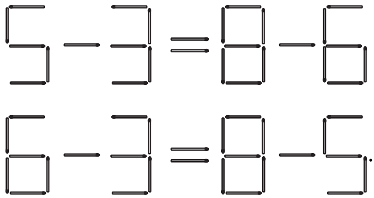
11.4. a) V + VII + VI = XVIII;

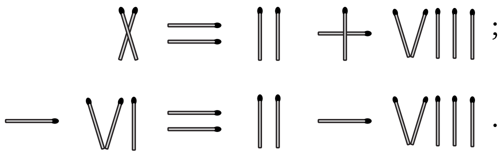
b) IX − V = IV;

c) XI − VI = VII − II.

11.5. a) ;

b) ;

c) 

11.6. 

11.7. Impossible. **11.8.** No to both questions. **11.13.** Impossible.
11.15. Only 191. **11.16.** Impossible. **11.17.** Impossible.
11.18. Not always. **11.19.** The second player wins.
11.20. Impossible. **11.21.** a) No; b) yes.
11.22. Yes. **11.25.** Yes. **11.26.** Impossible. **11.27.** 13.
11.30. Three children, each of whom owns three socks.
11.34. Yes. **11.36.** No. **11.37.** One bowl.
11.38. Impossible. **11.39.** 29 rows. **11.40.** 7. **11.42.** 0.
11.45. One can place 1, 2, 4, 8, 16, 32, and 64 coins in the purses.
11.46. Yes. **11.47.** 2021_3.
11.48. On the plate with the 61 g load, weights of masses 3 g and 27 g should be placed, and on the other plate, weights of masses 1, 9, and 81 g.
11.49. 1 kg, 3 kg, 9 kg, and 27 kg. **11.51.** 5 weights.
11.52. With base 8. **11.55.** No. **11.57.** Yes. **11.58.** No. **11.59.** No.
11.61. 1200. **11.62.** 29 matches. **11.63.** No. **11.65.** 5 teams.
11.68. Yes. **11.70.** Anton. **11.73.** No.
11.74. The player who took the third place won.

11.75.

	1st	2nd	3rd	4th	5th
1st	—	0	1	1	1
2nd	1	—	1/2	1/2	1/2
3rd	0	1/2	—	1	1/2
4th	0	1/2	0	—	1
5th	0	1/2	1/2	0	—

11.76.

	1st	2nd	3rd	4th	5th
1st	—	0	1	1	1
2nd	1	—	1/2	1/2	1/2
3rd	0	1/2	—	1/2	1
4th	0	1/2	1/2	—	1/2
5th	0	1/2	0	1/2	—

11.77. 5. **11.78.** 6. **11.79.** 12. **11.80.** Yes. **11.81.** a) Yes; b) no.
11.82. Possible. **11.83.** 11. **11.84.** 54.
11.86. 60 envelopes in 40 s, 90 envelopes in 10 s.
11.87. 24 girls. **11.88.** 5 girls. **11.89.** 2. **11.90.** 1.
11.93. Vassya. **11.94.** Petya. **11.95.** $\frac{1}{2}$.
11.96. Most likely there are three black balls and one white ball in the box.
11.97. Yes. **11.99.** Wednesday. **11.100.** Yes.
11.101. 160, 900, 936, or 972.
11.102. a) 31 coins; б) 30 coins.

Solutions

Chapter 1. Natural Numbers

1.1. The last digit is not changed under multiplication by 5, hence it is either 0 or 5. Therefore, the required number is either $0 \cdot 5 = 0$ (but this is not a natural number) or $5 \cdot 5 = 25$.

1.2. *First solution.* It is easy to check that if the last digit does not change under multiplication by 7, then it is either 0 or 5. Hence the required number is is either $0 \cdot 7 = 0$ (but this not a two-digit number) or $7 \cdot 5 = 35$.

Second solution. If $10a + b = 7b$, then $10a = 6b$, hence $a = 3$ and $b = 5$.

1.3. *First solution.* The number is divisible by 5, hence the last digit is either 0 or 5. The last digit cannot be 0. If $10a + 5 = 5(a + 5)$, then $5a = 20$ and $a = 4$.

Second solution. If $10a + b = 5a + 5b$, then $5a = 4b$, hence $a = 4$ and $b = 5$.

1.4. If $10a + b = 3a + 3b$, then $a = 2$ and $b = 7$.

1.5. If at least two digits of the number are different, one could exchange their places, so the solution is not unique. It follows from the condition that all the digits cannot be zeros. If all the digits are equal, greater than 1, but less than 9, it is always possible to increase one digit by 1 and decrease another by 1, yielding another solution. Only in the case when the neighbour is 45 years old and the number of the ticket is 99 999 is the solution unique.

1.6. The only natural numbers that possess this property are the numbers from 1 to 62 and from 100 to 999.

1.7. See the answers section.

1.8. Among the numbers from 1 to 100, exactly 20 contain the digit 1, namely, the number 1 itself, ten numbers from 10 to 19, the number 100, and the numbers $21, 31, \ldots, 91$ (there are eight). Hence none of these numbers was erased. There are exactly 19 numbers containing the digit two, namely the number 2 itself, the numbers $20, 21, \ldots 29$, as well as the numbers $12, 32, 42, \ldots, 92$ (there are eight of them), Hence Mike did not erase any of those, either. There are $19 + 20 - 2 = 37$ such numbers in all (we subtracted 2 above, because the numbers 12 and 21 were counted twice). Thus $37 + 30 = 67$ numbers remained, so Mike erased $100 - 67 = 33$ numbers.

1.9. First we can put in a digit between the ones and consider the numbers $121{*}2$, $131{*}{*}3$ and $141{*}{*}{*}4$ (there is no need to consider the same expressions written in inverse order). In the first and third cases it is impossible to choose digits in the required way, in the second case we obtain the number 41 312 432. The same digits may be written in inverse order.

1.10. The sum of digits of the required number is equal to the number of its digits, i.e., it equals 10. Knowing the nonzero digits of the required number, it is possible to indicate the number itself. For instance, such number with digits 9 and 1 must also have 8 zeros, i.e., it equals 1 000 000 018. But this number does not have the digit 9, so it does not work. The required number containing the digit 8 cannot contain another digit 8, hence its eighth digit is 1, hence its nonzero digits are 8, 1, and 1, the number itself is 1 000 000 107, which also does not work. For the digit 7, we can similarly work out two variants ($7, 1, 2$ and $7, 1, 1, 1$), for the digit 6, three variants ($6, 1, 3$; $6, 1, 1, 2$ and $6, 1, 1, 1, 1$). Now for the digit 7, there must be 7 zeros, i.e., three nonzero digits, and for 6, four of them. To the three digits $7, 1, 2$ corresponds the number 1 100 001 007, it also does not work. But now to the numbers $6, 1, 1, 2$ corresponds the number 2 100 010 006, and this works.

There are no other numbers with the required properties, because such a number must have the digit 6, 7, 8, or 9. Indeed, if none of these digits appear, then the number has no more than 5 zeros, and so it has 5 nonzero digits. The number has the digits that correspond to the places where these digits stand. The first nine places cannot be filled by zeros and ones, therefore the sum of digits of the number is more than $1 + 2 + 3 + 4 = 10$, which is impossible.

1.11. The required number is the number from Problem 1.10 with the last digit moved to the first place.

1.12. The sum of digits of each of the required numbers equals the number of digits of the other one, i.e., it is equal to 10. If we know the nonzero digits of the first number, we can find the second number, and from the latter we can recover the first number. For instance, the first number with digits 9 and 1 must also have eight zeros, then the second number will be 8 100 000 001. Hence the first number will be 7 200 000 010, but it has the wrong digits. Together with the digit 8, the first number will contain the digit 2 or the digits 1 and 1. A similar verification shows that both variants don't work. Together with the digit 7, the first number will contain either the digits 1 and 2 or three digits 1.The first variant doesn't work, but the second finally yields the correct answer 7 101 001 000 and 6 300 000 100.
1.13. The first digit of the obtained number cannot be more than 6, the last one cannot be more than 8. Therefore, no number greater than 68 can be obtained.
1.14. See the answers section.
1.15. *First solution.* If the oldest sibling has more brothers than the youngest one, then the eldest child is a girl (and the youngest, a boy).

Second solution. Among the siblings, there are 4 brothers and 3 sisters. Hence the child who has 2 sisters and 4 brothers is a girl.
1.16. Nick has 7 more classmate boys than classmate girls, so there are 8 more boys than girls in the class. Moreover, there are twice as many boys than girls. So there are 16 boys and 8 girls.
1.17. The first weighing implies that 2 balls and 4 cubes weigh as much as two cars, while the second one implies that 2 balls and 4 cubes weigh as much 1 car and 5 cubes. Hence one car weighs as much as 5 cubes.
1.18. There are as many barefoot children as there are barefoot girls and barefoot boys taken together. There are as many girls as there are barefoot girls and girls with shoes on. There as many barefoot boys as there are girls with shoes on. To these equal numbers we add the number of barefoot girls, thereby obtaining equal numbers.
1.19. Let us give one chocolate to each of the children. The condition implies that that we have enough chocolates to give one more to each of the boys, but not to each of the girls. So there are more girls than boys. Now let us give one more one chocolate to each of the boys and take all the chocolates away from the girls. Since there are more girls than boys, these chocolates will be enough to give one more (a third one) to each of the boys.
1.20. The more flags there are to the right of a child, the further to the

left is his position. To the right of Maxim there is at least one pupil (otherwise there would not be any flags to his right). But all the pupils, except Dasha, necessarily stand further to the left than Maxim. Thus, Dasha stands to the right of Maxim, and holds 8 flags.

1.21. At the first position, we can place any of the three digits, at the second position, any one of the remaining two, at the third, the last remaining digit. Thus we obtain $3 \cdot 2 = 6$ numbers.

1.22. At the first position, we can place any one of the digits 2 and 5, at the second position, any one of the remaining two, at the third, the last remaining digit. Thus we obtain $2 \cdot 2 = 4$ numbers.

1.23. At the first position, we can place any one of the three digits 2, 5, and 7, at the second position, any one of the remaining two, at the third, the last remaining digit. Thus we obtain $3 \cdot 3 \cdot 2 = 18$ numbers.

1.24. Similarly to Problem 1.23, we obtain $4 \cdot 3 \cdot 2 = 24$ numbers.

1.25. Let us take one pill away from Rhinoceros, two from Hippopotamus, and three from Elephant. Then all three animals will have the same number of pills, and since 2000 pills remain, each of them will have 500 pills. We took 3 pills away from Elephant, therefore Doctor Dolittle prescribed him 503 pills.

1.26. Let us give Mary two balloons. Then she will have as many balloons as Dolly. Now let us take one balloon away from Alex. Then everyone will have as many balloons as Dolly. The three of them will have a total of $11+2-1 = 12$ balloons. Therefore, Dolly has 4 balloons.

1.27. *First solution.* Let us add one sister. Then there will be an equal number of brothers and sisters. If we now remove two sisters, there will be twice as many brothers than sisters. Hence two sisters is as many people as half the number of brothers, so there are four brothers.

Second solution. Any sister has half as many sisters than brothers. Suppose two brothers come up to each sister. We then obtain several triplets and one solitary sister. Now suppose one brother in each triplet moves away from it. If one of them joins the solitary sister, then only one brother will remain (since the number of brothers is greater by one than the number of sisters). So there are two triplets, and therefore 4 brothers and 3 sisters in the family.

Third solution. Suppose there are n brothers and m sisters. Then $n-1 = m$ and $n = 2(m-1)$. Hence $m+1 = 2m-2$, i.e., $m = 3$.

1.28. *First solution.* If one removes 3 sweets from the right pocket, there will be an equal number of candies in the two pockets, i.e., $(27-3) : 2 = 12$ in each. Initially there were half as many sweets in the left pocket, namely $12 : 2 = 6$ sweets.

Second solution. After the sweets have been moved, the number of them in the left pocket doubled, and the number of sweets in the right pocket became equal to twice the number of sweets in the left pocket plus three. Hence the total number of sweets in Johnny's pockets equals four times the number of candies initially contained in his left pocket, plus 3. Hence initially there were $(27 - 3) : 4 = 6$ sweets in his left pocket.

1.29. There are 5 brothers and one sister in the family.

1.30. Among the three numbers, there are two of the same parity; their sum must be even.

1.31. Suppose that all but two boys mounted — one to each donkey. Now let's put the boys on donkeys in twos in the following way:

1) one of the boys leaves his donkey and mounts another donkey as a second rider, so one donkey is left without a rider, as required;

2) the two boys who originally were left without a donkey join two others as second riders.

Result: three donkeys are "occupied", and one is not. Hence the number of boys is 6, and of donkeys, 4.

1.32. If January 1, 2, or 3 is a Friday or a Monday, then January 29, 30, or 31 is the fifth Friday or the fifth Monday in January. Hence January 1, 2, and 3 must be Tuesday, Wednesday, and Thursday, respectively.

1.33. The clocks chime simultaneously every 6 seconds from the beginning until they stop. Between the simultaneous chimes, they chime separately three times. At the 18th second, the fourth simultaneous chime will sound, and the clocks will chime separately 9 times.

1.34. Today is January 1st, and Nick turned 11 on December 31 last year.

1.35. Only the numbers 467 and 2345 can be neighbours of 36, while only the numbers 57 and 2345 can be neighbours of 590. Hence we must have the segment $57, 590, 2345, 36, 467$ (or in reverse order). It is then easy to adjoin the numbers 14, 27, and 178 to it : between two of these numbers only the number 178 can be placed.

Chapter 2. Operations with Natural Numbers

2.1. Subtracting one of the summands from the sum of the two numbers, one gets the other summand. Hence the second summand equals 2000, while the first is 20.

2.2. Both numbers equal $101 \cdot 1001 \cdot 999$.

2.3. That number of times is $(999 - 9) : 99 = 990 : 99 = 10$.

2.4. The sum of the subtrahend and the difference is equal to the minuend, hence the sum of the minuend, the subtrahend, and the difference is equal to twice the minuend.

2.5. The quotient is five times less than the dividend, so the divisor equals 5. The quotient is 7 times greater than the divisor, so it equals $5 \cdot 7 = 35$.

2.6, 2.7. See the answers section.

2.8. a) Assume that all the digits of the sum are odd. In the middle (third) position identical digits are added, hence a carry occurred from the second to the third position. Therefore, the sum of digits in the second (and the fourth, to which it is equal) position is greater than 10, hence there was a carry from the fourth to the fifth position. Thus the sum of digits in the fifth (and first) position is even. But by assumption, the sum of digits in the first position is odd. Contradiction.

b) Assume that all the digits of the sum are odd. In the ninth (middle) position identical digits are added, hence a carry occurred from the eighth to the ninth position. Hence the sum of digits in the eighth (and the tenth, to which it is equal) position is greater than 10, so there was a carry from the tenth to the eleventh position. Therefore, the sum of digits in the eleventh (and the seventh) position is even. Continuing similarly, we see that there was a carry over from the sixteenth position to the seventeenth. But the sum of digits in the seventeenth (as well as in the first) position is odd. A contradiction.

2.9. a) For instance, $219 + 912 = 1131$.

b) For instance, we can write eight pairs of digits 12. Then all the digits of the sum will equal 3.

2.10. Half of a half is one fourth. If a fourth of a number is equal to $1/2$, then the number itself is 2.

2.11. Four pikes constitute half of the catch.

2.12. The weight of two thirds of the brick is 2 kilograms.

2.13. Two parts of the same duration remain. So in a day there are three such parts.

2.14. When the passenger fell asleep, he still had two thirds of his trip to go. When he woke up, half of the distance, i.e., one third of the trip, was is still ahead. Another third of the trip he had travelled before he fell asleep. So he slept for one third of the trip.

2.15. By the condition, there were one and a half times more candies of the second quality than candies of the first quality. Mixing 2 pounds of candies of the first quality with 3 pounds of the second quality, we obtain 5 pounds that cost $3 \cdot 2 + 2 \cdot 3 = 12$ rubles.

2.16. Since there are 4 times more pencils in the second box that in the first one and 5 times more in the third box that in the first one, the total number of pencils is $1 + 4 + 5 = 10$ times more than the number of pencils in the third box. So there are $80 : 10 = 8$ pencils in the first box.

2.17. After the water is poured, there will be one part of the water in the first barrel and three parts in the second one. In all there are 4 parts and they constitute the same total of 80 qt. Hence, before the water was poured, there were 30 qt. in the first barrel and 50 qt. in the second one.

2.18. Let us take the number of slices of the orange that Sparrow got as one part. Then Hedgehog got two parts, Beaver got five parts. Beaver got four more parts than Sparrow, which constitutes 8 slices. Therefore, one part consists of 2 slices. So there are 8 parts and 16 slices.

2.19. Since Lisa and Bart ate an equal amount of pastry, and Lisa eats thrice as fast as Bart, the latter needed three times as much time as Lisa to eat the pastry. Since they started and ended simultaneously, Lisa ate jam for the same amount of time that Bart ate pastry (and *vice versa*). Hence Lisa ate jam three times longer than Bart and three times faster, so she ate 9 times more jam than Bart.

2.20. There are 20 pencils, and the blue and green ones together constitute seven parts. So there must be 6 or 12 blue pencils. Then there will be 1 and 13 or 2 and 6 green and red ones, respectively. By the condition, there are less red pencils than blue ones, so the first variant is impossible.

2.21. For the book "Three Little Pigs–2" each author must get one fourth of the royalties. But one of the pigs has already taken his share, hence Wolf should be given one third of the remaining money. For the book "Little Red Riding Hood–2" Wolf should also get one third of the money. So he must take one third of the total sum handed to him.

2.22. The first condition means that the number of goldfish is greater by 1 than a third of the initial quantity of all fish, the second, that there are 4 redfish less than one third of all fish. So there are 3 silverfish more than one third of all fish.

2.23, 2.24, 2.25. See the answers section.

2.26. The sum does not depend on the order of summands, so we can assume that they are written in decreasing order. Five digits can be divided into the following five groups:

$$\begin{array}{ll} 4+1 & (7777+7=7784), \\ 3+2 & (777+77=854), \\ 3+1+1 & (777+7+7=791), \\ 2+2+1 & (77+77+7=161), \\ 2+1+1+1 & (77+7+7+7=98), \\ 1+1+1+1+1 & (7+7+7+7+7=35). \end{array}$$

2.27. Let us replace the stars from right to left. First note that

$$1+2+4+8+16+32=63<64.$$

Hence there must be a plus sign in front of 64 because otherwise we will be subtracting 64 from a number which is less than 64, and the result of this subtraction cannot be equal to 27. We now have the equation

$$1*2*4*8*16*32+64=27.$$

There must be a minus sign in front of 32, otherwise the left-hand side will be too big, even if we replace all the remaining stars by minuses. Continuing in a similar way, we obtain

$$1-2+4+8-16-32+64=27.$$

2.28, 2.29, 2.30, 2.31. See the answers section.
2.32. For any choice of brackets, after the representation of the resulting expression as a fraction, the number 1 will be in the numerator, while the number 2 will be in the denominator. Moreover, the numbers 9 and $3 \cdot 6$, 5 and 10, 4 and 8, must be on opposite sides of the fraction bar. Therefore we obtain three possible variants of the representation of the number 7:

$$7=\frac{1\cdot 7\cdot 8\cdot 9\cdot 10}{2\cdot 3\cdot 4\cdot 5\cdot 6}=\frac{1\cdot 3\cdot 5\cdot 6\cdot 7\cdot 8}{2\cdot 4\cdot 9\cdot 10}=\frac{1\cdot 3\cdot 4\cdot 6\cdot 7\cdot 10}{2\cdot 5\cdot 8\cdot 9}.$$

These expressions correspond to the choice of brackets indicated in the answer section.
2.33. *First solution.*

$$1+2+3+\ldots+100=(1+100)+(2+99)+\ldots+(50+51)=50\cdot 101.$$

Second solution. Add the following two expressions term by term:

$$1+2+3+\ldots+100 \quad \text{and} \quad 100+99+98+\ldots+1.$$

2.34. *First solution.*

$$3+4+5+\ldots+49=(3+49)+(4+48)+\ldots+(25+27)+26=52\cdot 23+26.$$

Second solution. Add the following two expressions term by term:

$$3+4+5+\ldots+49 \quad \text{and} \quad 49+48+47+\ldots+3.$$

2.35. The total mass of the fish equals $((100+1900)\cdot 19):2=19\,000$ g. Hence each fisherman will get 1900 g.

2.36. The sum $1+2+3+4+5$ equals 15. Hence the first of the required numbers is greater than 1 by $\dfrac{875-15}{5}=172$.

2.37. Consider the six smallest natural numbers $1, 2, \ldots, 6$. Their sum is 21. Hence the desired equation will hold if one of the numbers is increased by 1. But if we increase one of the numbers from 1 to 5, then we will obtain two equal numbers. Thus we must increase the last number, i.e., take 7 instead of 6. As a result, we obtain the required collection: $1, 2, 3, 4, 5, 7$.

2.38. For the digit 0 in the unit position, the digit in the ten position can assume nine values (from 1 to 9), for the digit 1 in the unit position, eight values (from 2 to 9), ..., for the digit 8 in the unit position, only the value 9. In all we obtain

$$1+2+3+\ldots+9=45 \text{ numbers.}$$

2.39. The required expression equals

$$25\cdot(26-24)+23\cdot(24-22)+21\cdot(22-20)+19\cdot(20-18)+$$
$$+17\cdot(18-16)+15\cdot(16-14)=2(25+23+21+19+17+15)=$$
$$=2(40+40+40)=2\cdot 120.$$

2.40. A three-digit number that has the digit a in the hundred position, the digit b in the ten position, and the digit c in the unit position equals $100a+10b+c$. Looking through all the nine three-digit numbers, we notice that each digit appears in them once in each of these three positions. Thus each digit will occur once in the sum of the nine numbers with coefficient 100, once with coefficient 10, and once with coefficient 1. Therefore the required sum does not depend on the order in which the numbers are written, and equals

$$(100 + 10 + 1)(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 111 \cdot 45 = 4995.$$

2.41. From the addition in the unit column, we see that $\mathrm{B} \neq 0$. Therefore, the addition in the ten column shows that $\mathrm{B} = 9$. Then the addition in the hundred column shows that $\mathrm{A} = 4$. Returning to the unit column, we find $\mathrm{C} = 5$.

2.42. Clearly, $\mathrm{B} = 1$. In the addition $\mathrm{B} + \mathrm{B}$, we can get either 2 or 3 (if there is a carry from the ten column). But the digit C is even (being the last digit of sum $\mathrm{D} + \mathrm{D}$). Hence $\mathrm{C} = 2$. In the addition $\mathrm{D} + \mathrm{D}$, the last digit will be 2, hence D is 1 or 6. But the digit 1 is already occupied by B, so $\mathrm{D} = 6$.

2.43. Clearly, $\mathrm{V} = 1$. The digit Z is even since it comes from the addition $\mathrm{W} + \mathrm{W}$ and is greater than 4 since in the addition $\mathrm{Z} + \mathrm{Z}$ a carry occurs. Hence $\mathrm{Z} = 6$ or 8. If $\mathrm{Z} = 8$, then $\mathrm{W} = 6$ or 7 and the last digit of the number $\mathrm{W} + \mathrm{W}$ is 8. This is impossible, so $\mathrm{Z} = 6$. So, $\mathrm{W} = 2$ or 3 and the last digit of the number $\mathrm{W} + \mathrm{W}$ is 6, hence $\mathrm{W} = 3$. Thus $\mathrm{Y} + \mathrm{Y} = 16$ and $\mathrm{Y} = 8$. For the remaining digits $2, 4, 5, 7$, and 9, the equality $\mathrm{X} + \mathrm{X} = \mathrm{U}$ will hold only if $\mathrm{X} = 2$ and $\mathrm{U} = 4$.

2.44. Clearly, $\mathrm{M} = 1$. The digit H is even (it comes from the addition $\mathrm{L} + \mathrm{L}$) and greater than 4 (in the addition $\mathrm{H} + \mathrm{H}$ a carry occurs). Hence $\mathrm{H} = 6$ or 8. If $\mathrm{H} = 6$, then $\mathrm{K} = 2$ or 3, hence $\mathrm{I} = 4$, 5, 6, or 7. Then in the addition $\mathrm{I} + \mathrm{I}$, we get 1 only if $\mathrm{I} = 5$ and $\mathrm{K} = 2$. We come to a contradiction because in the addition $\mathrm{I} + \mathrm{I}$ a carry occurs. Therefore $\mathrm{H} = 8$. So $\mathrm{K} = 6$ or 7. In the addition $\mathrm{K} + \mathrm{K}$, we get the last digit I, while in the addition $\mathrm{I} + \mathrm{I}$, we get the last digit 1, so that $\mathrm{K} = 6$ or 7. Besides, $\mathrm{L} + \mathrm{L} = 18$, hence $\mathrm{L} = 9$. For N and J only the digits 2, 3, 4, and 6 remain. We know that $\mathrm{N} = 2 \cdot \mathrm{J} + 1 - 10 = 2 \cdot \mathrm{J} - 9$. Therefore $\mathrm{N} = 3$ and $\mathrm{J} = 6$.

2.45. Multiplying Q by 4, we obtain a one-digit number, so Q is either 1 or 2. Besides, Q is even. Hence $\mathrm{Q} = 2$ and $\mathrm{T} = 8$. Multiplying R by 4, we obtain a one-digit number, so R is 0 or 1 (2 is already taken). If $\mathrm{R} = 0$, then $4 \cdot \mathrm{S} + 3$ ends in 0, which is impossible. Hence $\mathrm{R} = 1$ and $4 \cdot \mathrm{S} + 3$ ends in 1, i.e., $4 \cdot \mathrm{S}$ ends in 8. Two is already taken, so $\mathrm{S} = 7$.

2.46. Assume that $\mathrm{Z} = 9$. Multiplying 59 by 5, we obtain 295, hence $\mathrm{X} = 9$, which is impossible. Similarly $\mathrm{Z} = 8$, 6, and 5 will give identical digits for different letters. Only $Z = 7$ remains. Multiplying 57 by 5, we find $X = 8$. Now multiplying 857 by 5 and so on, we obtain $W = 2$ and $V = 4$.

REMARK. Multiply 142 857 by $2, 3, \ldots, 6, 7$ and see what happens.

2.47. It is clear from the rebus that $A + C = 10$, $A + B + 1 = C + 10$ (because a carry occurs) and $A + 1 = B$. It follows from the last two equations that $2 \cdot A + 2 = C + 10$. Adding this equation to the first one, we obtain $A = 6$. Hence, $B = 7$ and $C = 4$.

2.48, 2.49, 2.50. See the answers section.

2.51. Paul obtained that a number ending in GG under addition with a number ending in 11 gives a number whose last two digits are different. This is possible only if $G = 9$. Then $H = 0$ and $E = 1$.The number GHIEH begins with the digit 9, hence $D = 8$. Thus, 1 and 2 are added in the position of thousands, and this addition causes no carry, a contradiction.

2.52. Since $R > N$, we have $R > 1$. Since we are looking for the smallest number, let us take $N = 1$, $R = 2$, $O = 0$. Then $M \geq 3$. The case ROOM = 2003 is possible: $35 + 1968 = 2003$ or $38 + 1965 = 2003$.

2.53. Let us express the original number as $6ABC$, and choose the digits A, B, and C so as to satisfy the equation $6ABC - ABC6 = 1152$. Clearly, we have $C = 8$ and $6AB8 - AB86 = 1152$. Now it is easy to see that $B = 3$ and $6A38 - A386 = 1152$.Therefore, $A = 5$.

2.54. Denote the digits used to construct the numbers in increasing order: $A \leq B \leq C \leq D$. The smallest number consisting of these digits is $ABCD$, while the largest is $DCBA$. Choose digits A, B, C, and D so as to have

$$ABCD + DCBA = 11\,990.$$

Clearly $A+D = 10$, hence there was a carry from the position of hundreds to the position of thousands, i.e., $C+B > 9$. Now looking at the position of tens we see that $B + C = 18$. Hence $B = C = 9$. By condition, the number is the largest of the possible ones, hence $D = 9$ and $A = 1$.

2.55. Some missing digits are easy to recover:

$$\begin{array}{rcccc}
\times & & 3 & * & 5 \\
 & & & 4 & 1 \\ \cline{3-5}
 & & 3 & * & 5 \\
+\ 1 & 2 & * & 0 & \\ \hline
1 & * & * & * & 5
\end{array}.$$

In the number $3*5$, the star can only be zero or one since $325 \cdot 4 = 1300$.

2.56. First let us indicate the digits that are easy to find:

$$\begin{array}{cccccc}
 & \times & * & 2 & * & 3 \\
 & & & & * & * \\
\hline
+ & * & * & * & 8 & 7 \\
* & * & * & * & 6 & \\
\hline
2 & * & 0 & 0 & 4 & 7
\end{array}.$$

Now it is easy to see that the second factor is 29: only by multiplying 3 by 9 can one get the last digit 7, and only by multiplying 3 by 2 can one get the last digit 6. The product of 3 and 9 is 27, hence the digit 8 is obtained by adding 2 and 6. Under the multiplication by 9 of the digit between 2 and 3, we obtain a number ending in 6, hence that digit is 4. Now we can write:

$$\begin{array}{cccccc}
 & \times & * & 2 & 4 & 3 \\
 & & & & 2 & 9 \\
\hline
+ & * & * & 1 & 8 & 7 \\
* & * & 4 & 8 & 6 & \\
\hline
2 & * & 0 & 0 & 4 & 7
\end{array}.$$

In front of the one, we have the digit 5; it is obtained by adding 2 and 3. Here the digit 3 can only be obtained by multiplying 9 by 7, so 7 is the first digit of the first factor.

2.57. It follows from the position of the stars that the second and fourth digits of the quotient are zeros. The divisor under multiplication by 8 yields a two-digit number, and under multiplication by the first and last digits of the dividend, it yields three-digit number, so the first and last digits of the quotient are nines, so the quotient is 90 809. The divisor is a two-digit number that under multiplication by 8 gives a two-digit number, and under multiplication by 9, a three-digit one. Hence the divisor is 12. Knowing the divisor and quotient, it is easy to recover the rest.

2.58. See the answers section.

2.59. The sum of the numbers from 1 to 9 is 45. If the sum in each of the four triplets is 15, then summing these four sums one obtains 60. This is the sum of all the numbers from 1 to 9, but with number in the centre counted three extra times (4 times instead of once). Hence, this number equals $(60 - 45) : 3 = 5$. The remaining numbers must be just split into pairs that give 10 when summed.

2.60. The sum of the numbers from 1 to 6 is 21. If the sum of each of the triplets is 10, then, summing these three sums, one obtains 30. Here the numbers at the vertices were counted twice, hence their sum equals $30-21=9$, while the sum of the numbers at the sides is $21-9=12$. Using the fact that $1+3+5=9$ and $2+4+6=12$, we can first try to place 1, 3, and 5 at the vertices, and then, 2, 4 and 6 at the sides. This quickly leads to success.

2.61. a) The sum of the numbers from 1 to 9 is 45, hence the sum of the numbers at the vertices of the triangle equals $17 \cdot 3 - 45 = 6$. These numbers can only be the numbers 1, 2, and 3. The sums of numbers at the sides are 12, 13 and 14, and now it is easy to fill the blanks. The answer on page 116 corresponds to the representations

$$12 = 4 + 8, \quad 13 = 6 + 7, \quad \text{and} \quad 14 = 5 + 9.$$

There are also other variants, e.g.,

$$12 = 5 + 7, \quad 13 = 4 + 9, \quad \text{and} \quad 14 = 6 + 8.$$

b) The sum of the numbers at the vertices of the triangle is $20 \cdot 3 - 45 = 15$. We can use the representations

$$15 = 2 + 5 + 8 \quad \text{and} \quad 15 = 4 + 5 + 6.$$

In the first case we immediately obtain:

$$7 = 3 + 4, \quad 10 = 1 + 9, \quad \text{and} \quad 13 = 6 + 7.$$

In the second case we get several variants:

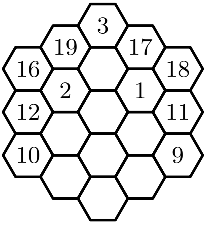
$$9 = 1 + 8 = 2 + 7, \quad 10 = 1 + 9 = 2 + 8 = 3 + 7, \quad 11 = 2 + 9 = 3 + 8.$$

From them it is easy to select those in which all the digits differ:

$$9 = 1 + 8, \quad 10 = 3 + 7, \quad 11 = 2 + 9.$$

2.62. The sum of the first three numbers is 20, and the sum of the 2nd, 3rd, and 4th numbers is also 20. Hence, the 4th is equal to the first one, i.e, to 5. Similarly, the 7th number equals the 4th, so it also equals 5. Now let us find the 6th number. It equals $20-8-5=7$. Then we easily find the other numbers.

2.63. The sum of the numbers from 1 to 19 is 190, the number of rows in each direction is 5, hence the sum of numbers in each row is $190 : 5 = 38$. Now we can indicate the numbers in several cells (see the figure). Then we can place the digit 7 under the cell with number 3 (by counting the sum of the other numbers in the diagonal rows containing that cell). After that 7 empty cells remain; into them the numbers 4,5,6,8,13,14, 15 must be placed. Consider the diagonal with numbers 10, 1, 18. The two empty cells in it by must be occupied by numbers whose sum is 9. They can only be 4 and 5. Now consider the diagonal with numbers 16, 2, 9. The two empty cells in it must be occupied by numbers whose sum is 11. They can only be 5 and 6. Hence 5 stands at the intersection of the two diagonals (i.e., in the centre of the hexagon), while the second numbers on the diagonals are 4 and 6. Now 4 empty cells remain. They must be occupied by the numbers 8, 13, 14, 15. The sum of the number in the lowest cell with the neighbouring cells (still empty at present) must equal 28 and 29, hence the lowest cell is occupied by the number 15. The remaining three cells are easily filled in.



2.64. Having in mind that we cannot put 3 ones, 3 twos, or 3 threes in a row, it is easy to find the solution: $1, 2, 1, 2, 2, 3, 3, 1, 1, 3$.

2.65. The sum of all the numbers from 1 to 11 is 66. Let us cut the pentagon into three triangles using two diagonals. The sum of all numbers in these triangles is 66, hence the sum of numbers in each triangle is 22.

Let x be the sum of numbers in the small triangles having a common side with the big pentagon, and let y be the sum of numbers in the small triangles having but a common vertex with the big pentagon. The sum of numbers in all the small triangles is $x + y$, and one number is in the small pentagon, hence $x + y \leq 65$.

Each of the five diagonals of the (big) pentagon cuts off a triangle from it. Let us write the sum of numbers in each of these five triangles and find the sum of these five sums. On one hand, this sum equals $5 \cdot 22 = 110$, on the other hand, it is $2x + y$. Hence,

$$2x + y = 110 \quad \text{and} \quad x = 110 - (x + y) \geq 110 - 65 = 45.$$

But $x \leq 7 + 8 + 9 + 10 + 11 = 45$. Hence 1 stands in the small pentagon while the numbers from 7 to 11 stand next to the sides of the pentagon, and the small triangles between these numbers are occupied by the numbers from 2 to 6. Let us fill in the remaining blanks by the numbers from 2 to 6, in the increasing order. After that in each of the big triangles the triangles two sides of which are diagonals of the pentagon, all the numbers save one will be known. The five remaining numbers are easy to find.

2.66. In the decimal expression of the number 10^1, there are two digits, in the expression of the number 10^2, three digits, and so on.

2.67. $2^{20} = (2^{10})^2 = 1024^2 = 1\,048\,576$.

2.68. They can: $2^{10} = 1024$, $2^{11} = 2048$, $2^{12} = 4096$, and $2^{13} = 8192$.

2.69. Let $2a$ be the smallest n-digit power of two. Then $a < 10^{n-1} < 2a$, hence

$$10^{n-1} < 2a < 4a < 8a < 8 \cdot 10^{n-1} < 10^n.$$

2.70. If $10^n > a > 10^{n-1}$, then $2^4 a = 16a > 10^n$.

2.71, 2.72, 2.73. See the answers section.

2.74. The sum of all the numbers from 1 to 9 is 45, the sum of all the numbers in each of the three rows is 3 times smaller.

2.75, 2.76. See the answers section.

2.77. Let us add the sums of numbers in the middle column, in the middle row, and in the both diagonals. The result equals $15 \cdot 4 = 60$. But this sum is the same as the sum of all the numbers plus three times the number in the centre. Therefore the number in the centre is 5.

2.78. In the magic square from Problem 2.76 subtract 1 from each number.

2.79. The missing numbers are 1, 7, 12, and 14. The sum of the missing numbers in the last row is 15, hence the missing numbers in that row are 1 and 14. The sum of missing numbers in the last column is 13, hence the numbers missing in it are 1 and 12. Therefore, the number in the corner is 1.

Chapter 3. Word Problems

3.1. The musketeers could have been riding:

a) towards each other;

b) away from each other;

c) in the same direction, Athos behind Aramis;

d) in the same direction Aramis behind Athos.

In these four cases the answers are:

a) $20 - (4 + 5) = 11$ leagues; b) $20 + (4 + 5) = 29$ leagues;

c) $20 + 5 - 4 = 21$ leagues; d) $20 - 5 + 4 = 19$ leagues.

3.2. The trip one way by bus takes $30 : 2 = 15$ min, hence the trip one way on foot takes 1 hr 30 min − 15 min = 1 hr 15 min. The trip back and forth by foot takes twice as much time, i.e., 2 hr 30 min.

3.3. Ken covered one extra half of the route. Hence Ken arrived later than Mary by half of the time spent in the whole route, i.e., by 10 minutes.

3.4. Every second the distance between them decreases by $8 - 7 = 1$ m.

3.5. Every day the distance between the messengers decreases by $45 - 40 = 5$ miles, hence the second one will catch up with the first in $40 : 5 = 8$ days.

3.6. *First solution.* If I had left 10 minutes after my brother, I would have caught up with him in 30 minutes (at the school). Five minutes is half of ten minutes, hence the distance I will have covered when I catch up with him is twice less, hence I will catch up with him twice faster.

Second solution. In 15 minutes after leaving I will cover half the route. At that moment, my brother will have been walking for 20 minutes, hence he will also have covered half the route.

3.7. Suppose that one hour after the pedestrian began walking, he reached the point C, while the cyclist reached the point D; the distance from A to C is equal to the distance from C to D. Fifteen minutes later, the pedestrian and the cyclist met at the point E. The pedestrian covered the route AC in 60 minutes, therefore he also covered CD in 60 minutes. So the pedestrian's route ED took up $60 - 15 = 45$ minutes, while the cyclist's route DE took up 15 minutes. Therefore, the speed of the pedestrian was three times less than that of the cyclist. We know that the cyclist covered the route BD in one hour, so the pedestrian will cover that distance in 3 hours. So, the pedestrian covered the route AD in 2 hours, and the whole route in 5 hours.

3.8. If Peter comes back home to get the pen, he will need $3 + 7 = 10$ minutes more than he would need to if he had not returned. This means that it required 10 minutes to cover the distance from home to the place when he remembered the pen and back. Therefore, Peter remembered the pen when he had walked 5 minutes; at that moment he had covered one fourth of the route.

3.9. Since the squirrel runs back twice slower, the time used for the return trip is twice what it needs for the trip from the nest to the tree.

Hence the time needed to go from the nest to the tree is three times less than the time needed for the trip back and fourth, i.e., it is $54 : 3 = 18$ s. Therefore, the distance from the nest to the tree is $18 \cdot 4 = 72$ metres.

3.10. On the first day, the snail will crawl up 4 metres, and then crawl down to the height of 2 m. On the second day, the snail will slide up to the height of 6 metres, and then slide down to 4 m. From that height the snail will reach the top of the post in one day.

3.11. The pedestrians will meet in one hour, therefore the fly, flying for one hour at 14 k/h, will cover 14 km.

3.12. At 60 k/h, the car covers exactly one kilometer in precisely one minute, so it is impossible to gain the entire minute.

3.13. Let us see which ships will meet the ship leaving from Le Havre. First of all, it is the 6 vessels on their way to Le Havre (one left 6 days ago, the next, 5 days ago, and so on, the last one of them on its way since one day ago). Secondly, in 7 days of the trip, 7 more vessels that have left from New York will meet our ship (one leaves at the same time as ours, the next, one day later, and so on, the last one of them one day before our ship arrives to New York. All together we get 13 vessels.

3.14. When Johnny saw the bus, he had covered one third of the route from one stop to the other. If he runs back now, he will get to the previous stop at the same time as the bus. Therefore, if Johnny runs one third of the route forward instead, then during that time the bus will reach the first stop. Having run forward the remaining third of the way, he will reach the next stop simultaneously with the bus, which will have covered the full distance between successive stops. Hence, the bus moves three times faster than Johnny, and so its speed is 60 km/h.

3.15. The frogs cover the distance of 120 cm simultaneously. But here they are to cover the distance of 20 m = 2000 cm twice. The number 2000 is divisible by 40, hence Grunty will cover exactly 4000 cm. But 2000 is not divisible by 60, hence Croaky will have to cover a bigger distance (namely, 2040 cm in each direction). Therefore, Grunty will win.

3.16. For the whole trip of the car (back and forth), it took 20 minutes less than expected, so the car met the truck 10 minutes before its scheduled arrival at the airport. Hence, the meeting of the car and the truck occurred 10 minutes before the scheduled landing time of the plane. The plane actually landed 30 minutes before the car and truck met, i.e., 40 minutes earlier than scheduled.

3.17. Suppose Ann walks slower than Boris, and Cyril, slower than either one of them. Then Boris will catch up with Ann on the first day, then the two of them will move at Ann's speed, and Cyril will not catch up with them. On the next day Ann can catch up with Cyril, and then they will continue at his speed. After that Boris, who walks faster than Cyril, will be able to catch up with Cyril.

3.18. Ian first meets cabin № 13, then cabin № 12, so the numeration is in the direction of the lift's motion. Let the distance between successive cabins be a unit of length. Then the distance along the cable between cabin № 42 and cabin № 12 is 69 units: 57 units to cabin № 99 plus 12 units to cabin № 12. The distance between Ian and the upper terminal is equal to half of that sum, namely 34.5 units. The speed of approach of two cabins moving towards each other is twice the speed of the lift. Hence, any cabin covers one unit of length in 30 s. = 0.5 min., so that Ian will reach the upper terminal in $34.5 \cdot 0.5 = 17.25$ minutes.

3.19. Suppose that at the moment when the bus was passing Peter, the motorbike was x km. away. This means that during the time Peter ate three spoonfuls, the motorbike covered x km. The car, moving twice as fast as the motorbike covered, during these three spoonfuls, $2x$ km. It covered the same distance during the time Peter ate the next three spoonfuls. So, when the bus was passing Peter, the car was four times further from him than the motorbike. Now when the bus passed Sam, both the car and the motorbike were $2x$ km away. Thus, while the bus was moving from Peter to Sam, the car caught up with the motorbike, having covered its lag of $4x - x = 3x$ km. During that time interval, the car decreased its lag from the bus by $4x - 2x = 2x$ km. This means that the speed at which the car catches up with the motorbike (30 km/h) is one and a half times the speed at which it catches up with the bus. Therefore, the car catches up the bus at the speed of 20 km/h, while the speed of the bus is $60 - 20 = 40$ km/h.

3.20. Consider the part of the stairwell that Dmitry overcame on foot. He moves down the same part using the lift. On one hand, the move up takes twice as much time as the move down, on the other hand, it is more by 10 seconds. This means that for this part of the way he will need 10 seconds by lift and 20 seconds on foot. The whole way by lift takes up 60 seconds, hence on foot Dmitry walked one sixth of the way. He walked up a whole number of distances between successive floors. Had this whole number been two or more, Dmitry would have used the lift for 10 floors or more and ended on the 12th floor or higher. But the house has only nine floors, hence Dmitry walked up 1 floor and rode by lift for 5. Hence, he lives on the sixth floor.

3.21. The father is 9 times older than Serge, hence the difference of their ages is 8 times Serge's age. Therefore, Serge is 5 and Ivan is 15.

3.22. Two years ago, Lisa was also 8 years older than Nadine. Then she was three times older than Nadine, i.e., older by two times Nadine's age. Hence Nadine was 4, while Lisa was 12. So Nadine is now 6 and Lisa is 14.

3.23. The father is 23 years older than the son, hence the son is 23.

3.24. The father is 24 years older than the son. At present, the son is three times younger than the father, so 24 years is twice the son's age, so the son is 12, while the father is 36.

3.25. Grandma's age is 12 times that of the granddaughter, so the sum of their ages is 13 times the age of the granddaughter.

3.26. It follows from the condition that Tina's age is 17 years and 7 months, while Mike's age is 17 years and 8 months.

3.27. Let my sister be x years old, then I am nx years old and grandpa is n^2x. So $nx + n^2x = 84$, i.e., $xn(n+1) = 84$. By condition, $x < 7$, hence $n(n+1) > 12$. Besides, $n(n+1)$ is a divisor of the number 84. Therefore, $n = 6$ and $x = 2$.

3.28. In 10 minutes 5 cats will catch twice as many mice.

3.29. Johnny eats 100 g of jam in 1 minute, while Davy eats 200 g a minute, hence together they eat 300 g in 1 minute.

3.30. In 4 days 10 beavers perform half of the work. The new beavers are to carry out the remaining half of the work in 2 days. Therefore one must call twice as many beavers.

3.31. In two days the lumbermen will saw off 6 logs, and during the third day, they will chop them up. In one day they will be able to saw off and chop up one third of these logs.

3.32. In 70 days the man can drink 5 casks of the drink alone and 7 casks together with his wife. Therefore, in 70 days, the wife alone can drink 2 casks.

3.33. *First solution.* If two (identical) dogs grab the sausage at its ends and take a bite, there will be 300 g of sausage left. If two cats grab the sausage at its ends and take a bite, there will be 500 g left. Hence two dogs and two cats will leave 800 of two sausages. And one cat and one dog will leave twice less of one sausage.

Second solution. The dog intends to bite off 300 g less, than what it leaves. Had he bitten off 150 g more, leaving 150 g less, he would get exactly half of the sausage. Similarly, the cat would get half of the sausage, had it bitten off 250 g more than it intended. In that case, the whole sausage would have been eaten, hence the remainder weighs $150 + 250 = 400$ g.

3.34. In one hour of work the fast digger shovels out more, but the two diggers are equally paid. Hence one metre of the tunnel dug out by the fast digger costs less. When the two diggers work together, the fast digger digs out more than half the tunnel, hence it is cheaper to dig from opposite ends.

3.35. 100 rubles constitute half the cost of the book.

3.36. The big bird costs twice as much as the small one, so 5 big birds and 3 small ones cost as much as 13 small ones, while 5 small ones and 3 big ones as much as 11 small birds. Thus the purchases differ by 2 small birds, and the difference in cost is 20 rubles. Therefore, one small bird costs 10 rubles, a big one, 20.

3.37. *First solution.* 3 cups and 3 saucers cost $25 \cdot 3 = 75$ rubles, hence one cup costs $88 - 75 = 13$ rubles.

Second solution. 4 cups and 4 saucers cost $25 \cdot 4 = 100$ rubles, hence one saucer costs $100 - 88 = 12$ rubles, while one cup costs $25 - 12 = 13$ rubles.

3.38. Suppose the student bought 11 notebooks and had 5 rubles left. Let's give him 7 rubles. Then he will have $5 + 7 = 12$ rubles, and for that money he will be able to buy $15 - 11 = 4$ notebooks, hence one notebook costs 3 rubles. So the student had

$$11 \cdot 3 + 5 = 38 \text{ rubles.}$$

3.39. Three doughnuts cost 9 pieces of cheesecake and 15 rubles. 10 pieces of cheesecake cost just as much. Hence one piece of cheesecake costs 15 rubles, while a doughnut costs $(15 \cdot 10) : 3 = 50$ rubles.

3.40. The profit from each customer is the same. Hence the difference $10 - 5 = 5$ rubles is equal to the wholesale price of two pens.

3.41. The customer shortchanged the store by 100 rubles. The store lost those 100 rubles.

3.42. The change using three different coins constitutes at least $1 + 2 + 3 = 6$ soldo. This is not enough for an icecream, so it costs at least 7 soldo. But the icecream cannot cost more than 7 soldo, because otherwise Pinocchio would have had at least $8+8+6 = 22$ soldo, whereas he had only one coin, i.e., no more than 20 soldo. Thus, one icecream costs 7 soldo. If Pinocchio had a 20 soldo coin, he would have received his change as one 13 soldo coin plus three 1, 2, and 3 soldo coins.

3.43. *First solution.* If all the 17 animals were ducklings, they would have $17 \cdot 2 = 34$ legs. The "extra" $44 - 34 = 10$ legs belong to the pups — two each. Therefore there are 5 pups.

Second solution. If the ducklings had 4 legs each, the total number of legs would be $17 \cdot 4 = 68$. There would be $68 - 44 = 24$ "extra" legs, twice as many as ducklings.

3.44. If all the animals were dogs, $10 \cdot 6 = 60$ biscuits would be needed. The extra 4 biscuits were needed because a dog eats one biscuit more than a cat. Therefore there were 4 cats and 6 dogs.

3.45. If each vehicle had 4 wheels, then $4 \cdot 45 = 180$ wheels would be needed, hence there would be $180 - 115 = 65$ extra wheels. To each sidecar motorbike corresponds one extra wheel, while to a motorbike without sidecar, two. But since there are twice as many motorbikes without a sidecar as motorbikes with one, the number of extra wheels is 5 times more than the number of sidecar motorbikes. We see that there were $65 : 5 = 13$ sidecar motorbikes, $13 \cdot 2 = 26$ motorbikes without a sidecar, and $45 - 13 - 26 = 6$ cars.

3.46. First of all, let us note that a chair with a person sitting on it has 6 legs, while a stool with a person sitting on it has 5 legs. Now various arguments are possible.

First solution. If the number of chairs and stools is 6 or less, then they have at most $6 \cdot 6 = 36$ legs, while if this number is 8 or more, then there are at least $8 \cdot 5 = 40$ legs. Hence the number of chairs and stools is 7. Seven stools with persons on them have $7 \cdot 5 = 35$ legs. The missing 4 legs belong to the chairs, so there are 4 chairs and 3 stools.

Second solution. The number of legs of stools with persons sitting on them is divisible by 5, hence it ends in the digit 0 or 5. But this number cannot end in 0, because then the number of legs of chairs with persons sitting on them would end in a 9, while this number must be even. Hence there can be 1, 3, 5, or 7 stools. A simple inspection shows that there are 3 stools and 4 chairs in the room.

3.47. The number of mushrooms held by Basil decreased by 10, while Helen's number increased by 10, and after that these numbers became equal. Hence the difference between these numbers was originally twice as much.

3.48. *First solution.* If Helen had no mushrooms at all, Basil would have given her half of his mushrooms. Hence he has to give her half of the "extra" mushrooms.

Second solution. The number of mushrooms held by Basil must decrease by as much as it must increase for Helen.

3.49. Suppose Ann gives half her mushrooms to Victor. Then all the children will have the same number of mushrooms, and this means that Victor had no mushrooms at all. Now, for Alex to get all of Ann's mushrooms, he must take Victor's and Ann's mushrooms. Then he will have the mushrooms of three children, namely Victor's, Ann's, and his own. The same number of mushrooms will be held by the other children, so there were three more children mushroom picking with Victor, Ann, and Alex.

3.50. Each ate 4 flapjacks, hence the first one ate all his flapjacks, while the second and third one shared the same number of flapjacks.

3.51. Each ate 5 flapjacks, hence the first gave one flapjack to the third one, while the second one gave him 4.

3.52. Since 6 mackerels weigh more than 10 herrings, all the more so they weigh more than 9 herrings. Dividing 9 by 3, we see that 2 mackerels weigh more than 3 herrings.

3.53. The conditions mean that 20 black cows together with 15 brown cows give as much milk in one day as 12 blacks cows and 20 brown cows. Hence 8 black cows give as much milk as 5 brown cows, so the brown cows give more milk per day than the black ones.

3.54. In the first two chests, the total number of gold coins is more by $7 + 15 = 22$ than the total number of silver coins. There is an equal number of gold and silver coins, hence in the third chest there are 22 gold coins less than silver ones.

3.55. The number of other white animals is 5 times that of white cats, while the number of cats is only 3 times more than the number of white cats.

3.56. After the first fisherman gave out his fish, the other fishermen will have $100 : 5 = 20$ fish. Hence, each caught at most 20 fish. Suppose that John caught exactly 20 fish. When fisherman handed out his fish, John got nothing, but everyone had the same number of fish. Hence when John leaves, the others will have 80 fish, at most 20 fish each, and each of them will be able to give out his entire catch to 4 others so that each should have the same number. It remains to prove that at least one of the fishermen caught exactly 20 fish. Indeed, if there is no such fisherman, then all the remaining fishermen have a total of at most $19 + 18 + 17 + 16 + 15 + 14 = 99 < 100$ fish, which is impossible.

3.57. If to the weight of Mary and her mother we add the weight of

Mary and her father, then we obtain $10 + 50 = 60$ kg more than the weight of the father and mother together. Hence the doubled weight of Mary equals 60 kg.

3.58. If we add the weights of Ken and his mother, of Ken and his father, of his father and mother, we get twice the total weight of all three, so they all together weigh $(100 + 120 + 140) : 2 = 360 : 2 = 180$ kg. If we subtract the weight of Ken and his mother from the total weight of all three, we obtain the father's weight: $180 - 100 = 80$ kg. The mother's weight is $140 - 80 = 60$ kg. Ken's weight is $100 - 60 = 40$ kg.

3.59. *First solution.* The weight of the milk in half the container is equal to the difference between the weight of the container filled with milk and the container half filled with milk, i.e., it equals 16 pounds. Hence, the weight of the container is 1 pound.

Second solution. The weight of the can is equal to the difference between the doubled weight of the container half filled with milk (i.e., the weight of the contents plus the doubled weight of the container itself) and the weight of the filled container (i.e., the weight of the container plus that of its contents). Therefore, the container weighs 1 kg.

3.60. Jeff earned the extra $17 - 5 = 12$ by hitting the targets, 2 shots for each hit. So there were 6 hits.

3.61. If Peter made 50 shots, then he must have earned 45 extra shots, so he must have hit the target 9 times. But he says that he hit it only 8 times. So he is wrong.

3.62. If all the questions are answered correctly, the participant gains $20 \cdot 12 = 240$ points. With any wrong answer, this score decreases by 22. The actual score is less than the maximum by $240 - 86 = 154$ points, hence the participant gave wrong answers to $154 : 22 = 7$ questions and answered correctly to the remaining 13.

3.63. If a log is sawed into pieces, there will be 1 piece more than cuts, because the first cut produces two pieces and each of the further cuts produces one more piece.

3.64. In order to saw a log in 5 pieces, one should make 4 cuts, and each cut takes 5 minutes.

3.65. When sawing one log, the number of pieces is 1 more than the number of cuts. Since there are 6 more pieces than cuts, 6 logs were sawn.

3.66. In order to cut a long log, four cuts must be made, to cut a short log, three. Hence the ratio of the number of long logs to that of short logs is three to four. Therefore 15 long logs and 20 shorter ones were given. The total number of cuts was $15 \cdot 4 + 20 \cdot 3 = 120$.

3.67. When a sheet is cut into 3 parts, the number of sheets increases by 2. The number of added sheets is $15 - 9 = 6$, hence $6 : 2 = 3$ sheets were cut.

3.68. Each gnome takes 1 square of cloth from the chest, and puts back 4 (i.e., adds 3 squares). Therefore, after the Seventh Gnome leaves, there will be $1 + 3 \cdot 7 = 22$ squares.

3.69. At each break, the number of pieces increases by 1, and 12 pieces must be obtained. Hence there must be 11 breaks.

3.70. The number of 5 metre logs is even but not four (because $42-20 = 22$ is not divisible by four). Hence we can saw either two five-metre logs and eight four-metre logs, or six five-metre logs and three four-metre logs. In the first case the number of cuts is $2 \cdot 4 + 8 \cdot 3 = 32$, in the second one, $6 \cdot 4 + 3 \cdot 3 = 33$.

3.71. The number of parts is always greater by 1 than the number of cuts. Thus, there are 4 red circles, 6 yellow circles, and 10 green ones. The total number of circles is $4 + 6 + 10 = 20$, so the number of parts, if one cuts along all the circles, is 21.

3.72. *First solution.* Adding a tee into a socket one increases the number of sockets by 2. Adding 21 tees to the original 8 sockets (in any order), one obtains $8 + 21 \cdot 2 = 50$ sockets.

Second solution. We can represent the sockets as logs and the addition of a tee as sawing of a log into three pieces. It is required to find the largest number of parts that can be obtained from 8 logs by sawing the logs no more than 21 times, each time into exactly three pieces. Each such cut increases the number of pieces by 2, hence there will be no more than $8 + 21 \cdot 2 = 50$ pieces.

3.73. Since the tourist doesn't care at what exchange office he will change his money, it follows that if the first office did not charge commission, the tourist would get for his rubles 7000 more soldos in the first office than in the second one. For each ruble in the first office the tourist would get 50 soldos more than in the second one. Therefore, he intends to exchange $7000 : 50 = 140$ rubles.

3.74. Mole can exchange 3 sacks of grain for 6 sacks of millet, then in his den there will be 11 sacks: 5 sacks of grain and 6 sacks of millet. In one month he will eat up 3 sacks of grain, and during the two other winter months, 1 sack of grain and 3 sacks of millet.

3.75. Pinocchio obtained 50 candies, so he performed exactly 50 transactions. He exchanged all the obtained euros back to dollars. Hence to each two transactions of the second type, three transactions of the first type correspond. In any set of two transactions of the second type and three transactions of the first type Pinocchio lost $2 \cdot 5 - 3 \cdot 3 = 1$ dollar. Thus he lost $50 : 5 = 10$ dollars.

3.76. The time in the first channel from the beginning of each part of the film to the beginning of next one is 22 minutes. During that time, on the second channel two 10-minute parts and two 1-minute ads will be shown. Therefore, to the beginning of each part on the first channel corresponds the same episode of the film on the second channel. When the last part on the first channel begins, there remains 20 minutes of the film and there will be no more ads, while the second channel will be showing two 10 minute parts with a 1 minute ad in between, so the film will end 1 minute earlier on the first channel.

3.77. Peter finished tenth while Basil finished in front of him, hence Basil finished ninth. Basil was fifteenth from the end, so 14 contestants finished after him. Therefore 23 people participated in the race.

3.78. Greg started first. In order for him to overtake Alex and Ellen 10 times, Alex and Ellen must have overtaken him at least 10 times. Since the total number of Alex's and Ellen's overtakes is $6 + 4 = 10$, it follows that they overtook only Greg and not each other. After Greg carried out all 10 of his overtakes, he turned out to be first again. Therefore the three runners finished in the same order as they started.

3.79. In 24 hours the clock moves ahead by 9 minutes, hence in 8 hours from 22:00 to 6:00 it will move ahead by 3 minutes, so that at 6:00 it will show 6:03. So the alarm should be set for that time.

3.80. In order to number pages 1 to 9, nine digits are needed; to number pages 10 to 99, one needs $90 \cdot 2 = 180$ digits. There remain 1203 digits, they can be used to number $1203 : 3 = 401$ pages. So the number of pages is $99 + 401 = 500$.

3.81. If the first strike was at the half hour, then the total number of strikes would be even. After the first strike at 1 pm there will be 8 strikes, if an hour is added, the number of strikes increases by 3. Hence the first strike was at 2, and the last at 4.

3.82. The total number of blueberry and strawberry muffins equals the number of raspberry muffins, while the number of blueberry muffins is less by 14 than the number of raspberry muffins. Hence there are 14 strawberry muffins. The total number of raspberry and blueberry muffins is twice that of strawberry muffins, i.e., $14 \cdot 2 = 28$, while the total

number of all the muffins is $28 + 14 = 42$. Hence there are $42 : 2 = 21$ raspberry muffins and $21 - 14 = 7$ blueberry muffins.
3.83. At the end there remained a total of $35 - 8 = 27$ yellow and white dandelions in the clearing, so there remained 18 yellow and 9 white ones. Therefore, at the beginning, there were $18 + 2 = 20$ yellow and 15 white dandelions $(15 = 35 - 20 = 9 + 8 - 2)$.
3.84. a) All the dandelions that were yellow the day before yesterday became white either yesterday or today. Hence there were $14 + 11 = 25$ of them.

b) From yesterday's yellow dandelions 11 turned white today, and the other $20 - 11 = 9$ will turn white tomorrow.
3.85. After visiting the homes of all his friends, Winnie will have $(1 + 2 + 3 + 4 + 5) - 5 = 10$ pots. Since before the last visit he already had 10, his last visit was to Tigger. He could have visited the other houses in any order, because the sum does not depend on the order of summands.
3.86. 5 crows flew away from the clearing, so 30 remained. Since the number of crows sitting on the birch became twice the number sitting on the alder, 20 crows remained on the birch and 10 on the alder. But before that 5 crows flew from the alder to the birch, hence originally there were 5 crows on the alder, while the other 30 crows were on the birch.
3.87. Dale collected $147 - 120 = 27$ nuts more than Chip. The amounts of nuts stored by the chipmunks for the winter must be the same, hence Chip must collect 27 nuts more than Dale, and this is 3 times what Dale must still collect. Hence Dale must collect 9 nuts, and his total amount of nuts will be $147 + 9 = 156$ nuts.
3.88. *First solution.* In each case the new number of bulbs is equal to the number of already planted bulbs plus the number of intervals between neighbouring bulbs; this number is smaller by 1 than the number of bulbs. Hence the number of bulbs after each step is less by one than the doubled number of already planted bulbs. After Doug had planted his bulbs, there were 113 of them. So, after Inna's work, the number of bulbs was $(113 + 1) : 2 = 57$. Similarly, after Tina's work, there were $(57 + 1) : 2 = 29$ bulbs, while Ken planted $(29 + 1) : 2 = 15$ bulbs.

Second solution. Suppose that the bulbs were not planted along a footpath, but in a circle. Then Tina will plant one extra bulb, Inna, two, Doug, four. Hence instead of 113 bulbs, there will be 120. On the other hand, in that situation after each planting the number of bulbs doubles. Therefore, Ken planted $120 : 8 = 15$ bulbs.

Chapter 4. Divisibility of Natural Numbers

4.1. For such an exchange to be possible, the number of red balloons must be even.

4.2. There exist precisely 8 tile ends (square halves of the tiles) with 5 spots: two on the double tile and six on the other tiles. In the line, all the fives occur in pairs. The five at the end of the line has no pair. Therefore, the remaining five is at the other end.

4.3. In the hands of each knight, the number written on the parchment changes parity. So after 33 changes, it becomes odd.

4.4. a) If the sum of two natural numbers is even, then either they are both even or both odd. In both cases their difference is even.

b) If the sum of two natural numbers is odd, then one is odd and the other is even. So their difference is odd.

4.5. The sum of an odd number of numbers can be represented as the sum of several pairs of numbers plus a separate number. The sum of two odd numbers is even. Hence the sum of several pairs of odd numbers is even. So when one odd number is added to that sum, an odd number is obtained.

4.6. The sum of page numbers on different sides of a sheet is odd. The sum of 25 odd numbers cannot be even.

4.7. If two girls stand together somewhere, then there are girls around them, and girls around the four girls, and so on. In that situation there are no boys, which contradicts the condition. We similarly come to a contradiction if we assume that two boys stand together. Hence boys and girls alternate, so there is an equal number of boys and girls. Therefore, there are five girls.

4.8. Consider the sum of the number of all participants in all 5 contests. This sum is odd, since an odd number of participants came to each contest. On the other hand, that same number can be obtained by adding, for each participant, the number of contests in which the participant took part. Since the sum is odd and all the summands are odd, the number of summands is odd.

4.9. If the number of Martians that have an odd number of hands were odd, then the total number of hands of all the Martians would be also odd. But when all the Martians hold hands, their hands can be counted by pairs, so the total number of hands of all the Martians is actually even. Thus, the number of odd-handed Martians cannot be odd, so it is even.

4.10. The sum of the written numbers is odd (it equals 21). At each step this sum increases by 2, so it remains odd. But the sum of six equal numbers is even.

4.11. If one of the numbers m and n is even, then the product mn is even. If both numbers are odd, then the sum $m + n$ is even. In both cases the product $mn \cdot (m + n)$ is even.

4.12. If the product of four integers is odd, then all four numbers are odd, and so their sum is even.

4.13. Suppose that there are no neighbouring numbers whose sum is even. Then the sum of any two neighbouring numbers is odd, which means that even and odd numbers alternate. But for an odd number of numbers, this is impossible.

4.14. When a girl leaves, the parity of number of shawls changes, hence when the 17th girl has left there will be an odd number of shawls on the hangers.

4.15. Even numbers do not influence the parity of the result, while the sum as well as the difference of two odd numbers is odd. In the relation there are 5 odd numbers. Hence for any choice of signs the result will be odd.

4.16. Denote the numbers by letters a, b, c, d, e. The number

$$a = (a + b + c) - (a + b + d) + (a + c + d) - 2c$$

is even. The fact that the other numbers are even is proved similarly.

4.17. Suppose that the construction is possible. Consider the ends (square halves) of the tiles with given number of spots (from zero to five). Now they occur 7 times each. Inside the constructed line of dominoes, they occur an even number of times. Hence at one of the ends of the line there must be a half-tile with the same given number of spots. This is true for each of the six possible numbers of spots, but the line has only two ends. Thus, a line of tiles with the required properties does not exist.

4.18. The number of digits in all the two-digit and one-digit numbers equals $9 + 90 \cdot 2 > 100$, hence the number of the last page of the book taken by Nicky must be two-digit. All the one-digit pages require 9 digits. That number is odd, and the addition of any number of pages with two-digit numbers adds an even number of digits, therefore the number of digits must remain odd.

4.19. The sum of the number of votes "for" plus the number of votes "against" is equal to the total number of MP's, and that number is even. Hence these numbers are either both even or both odd. Their difference cannot be odd.

4.20. The parity of the result does not depend on the choice of pluses and minuses, it only depends on the quantity of odd numbers in the original family of numbers. In the given case there are 10 of them, an even number, therefore the first player wins.

4.21. The number of all pairs of neighbours is N, and we know that half of them are enemy pairs. Let us divide the knights into groups of allied knights sitting next to each other. These groups alternate, hence their number is even. Each pair of enemy knights divides two such groups. Hence the number of pairs of enemy neighbours is equal to the number of groups, i.e., it is also even. The number N is twice the number of pairs of enemy neighbours, so it is divisible by 4.

4.22. Suppose this is not so. Then there can be no more than two boys sitting next to each other and next to any girl there is always at least one girl. Let us divide all the children into groups of girls sitting next to each other and boys sitting next to each other. These groups alternate, hence the number of groups of girls equals the number of groups of boys, and the total number of these groups is even. But there is no more than two boys in the groups of boys, and no less than two girls in the groups of girls. Therefore, all these groups consist of two people (otherwise there would be more girls than boys). So, there are 25 groups, an odd number. Contradiction.

4.23. The sum of 17 odd numbers is odd. Hence if the sum of 17 numbers is even, then there is at least one even number among them. Consider the first 17 of them and let us distinguish an even number among them.

The sum of the first 17 numbers is even while the sum of the first 18 is odd, so the 18th number is odd. The sum of numbers from the 2nd to the 18th is even, hence the first number has the same parity as the 18th, i.e., it is odd.

Now consider sequences of 17 and 18 numbers, starting from the 2nd. Similarly we find that the 19th number is odd and so is the 2nd. Let us continue this argument. Arriving at the distinguished even number, we will have to stop. The length of the sequence will be maximal when among its first 17 terms only the last one is even. This will be a 33-term sequence, in which the middle term is even and all the others are odd.

4.24. The numbers 10, 15, and 20 are divisible by 5, while the number 25 is divisible by 5^2. The other numbers from 10 to 25 are not divisible by 5. Hence the product under consideration is divisible by 5^5 but not by 5^6. It is also clear that it is divisible by 2^5.

4.25. The last digit of the product is 1, so the last digit of the difference is 0.

4.26. The price of the bottle (in cents) is divisible by 3, while the price of six boxes of matches is also divisible by 3, hence the total sum that Bill must pay is also divisible by 3. The barman demanded a sum which is not a multiple of 3, hence the sum was not calculated correctly.

4.27. If 10 sparrows eat up more than 1100 grains, then 9 sparrows eat more than $(1100 : 10) \cdot 9 = 990$ grains. We know that 9 sparrows eat up less 1001 grains, and among the numbers from 991 to 1000 only 999 is divisible by 9. Hence 9 sparrows will eat up 999 grains, and 1 sparrow, 111 grains.

4.28. The difference between a natural number and the number represented by its last two digits is divisible by 100, hence it is divisible by 4, too.

4.29. The last digit is 0, 4, or 8; the sum of digits is divisible by 9.

4.30. The number $a10a$ is divisible by 12 if and only if it is divisible by 4 and by 3. This means that a is divisible by 4 and $2a+1$ is divisible by 3. This is possible only for $a = 4$.

4.31. $234X2Y0$ seconds contain a whole number of hours, hence this number is divisible by 3600. Therefore, $Y = 0$ and the number $234X200$ is divisible by 9. Hence $2 + 3 + 4 + X + 2$ is divisible by 9, but this is possible only if $X = 7$.

4.32. If a number is divisible by 5, then its last digit is 0 or 5. Since the number must be divisible by 9, the sum of the introduced digits must equal 6.

4.33. The last digit is 0. The sum of the digits 8, 2, and 0 is 10. The sum of the next-to-last digit and 10 must be divisible by 9, hence the next-to-last digit is 8. The quotient is $8280 : 90 = 92$.

4.34. a) and b) These numbers are even, but are not divisible by 4.

c) This number is divisible by 3, but not by 9.

d) *First solution.* This number is divisible by 5, but not by 25.

Second solution. According to Problem 4.64, the square of a number ending in 5, ends in 25.

4.35. The last two digits are zeros. The number is divisible by 9, hence the fourth digit is 1.

4.36. The sum of digits of a number containing the same quantity of each of the digits is divisible by $0 + 1 + 2 + \ldots + 9 = 45$. So such a number is divisible by 3, while a power of two is not divisible by 3.

4.37. The number of peanuts originally in Victor's possession was a multiple of 3. After he gave Paul part of his peanuts, he was able to divide the remaining peanuts equally among the three squirrels. Hence the number of peanuts that he gave to Paul is divisible by 3 and no greater than 5. Thus Victor gave Paul 3 peanuts.

4.38. Each mouse can be at the storehouse one night with three other mice. In order to be at the storehouse with each of the 23 mice once, it needs $23 : 3$ nights. But 23 is not divisible by 3. So such a situation is impossible.

4.39. The digit 7 must correspond to one of the letters. Then the part of the relation that contains that letter will be divisible by 7, while the other part will not.

4.40. For instance, 444 444 is divisible by 33.

4.41. Consider, for instance, the numbers 1, 2, 3 or 2, 4, 6.

4.42. Peter took three times as many muffins as Johnny, while Tom took three times as many as Peter, i.e., 9 times as many as Johnny. Together they took $1 + 3 + 9 = 13$ times as many muffins as Johnny. Hence the total number of muffins taken is divisible by 13. But the only such number not greater than 15 is 13 itself. So, two muffins remained.

4.43. The number 8 can be represented as a sum of three distinct numbers in two ways: $8 = 1+2+5 = 1+3+4$. The numbers 1, 3, and 4 cannot be the three smallest divisors of the number A: if A is divisible by 4, it is also divisible by 2. Hence, the three smallest divisors of A are 1, 2, and 5. Thus, A is divisible by 10, but not by 4. Therefore, the number A ends in exactly one zero.

4.44. For the solution of each problem, the three girls together were given 7 chocolates (4, to the first, 2, to the second, 1, to the third). Hence the number of all chocolates earned must be divisible by 7, which is not the case for the number $20 \times 3 = 60$. Therefore, the girls made a mistake.

4.45. After each step, the number of shells decreases by 1, while the number of heaps increases by 1. Initially, there were 637 shells and one heap. Hence after n steps there will be $637 - n$ shells and $n + 1$ heaps. If each heap consists of three shells, one has $637 - n = 3(n + 1)$, i.e., $634 = 4n$. This is impossible, because 634 is not divisible by 4.

4.46. Use the fact that $16^{11} = (2^4)^{11} = 2^{44} = 2^5 \cdot 2^{39} = 32 \cdot 2^{39}$.

4.47. Use the fact that 111 is divisible by 37.
4.48. This number is even and divisible by 9 since it equals

$$(10^{23} - 1) + (10^{19} - 1) - 180.$$

4.49. Suppose that none of the 18 consecutive numbers is divisible by 18. Then they are all located between two consecutive multiples of 18. But between two consecutive multiples of 18 there are only 17 consecutive numbers. This contradiction shows that one of the numbers in question is divisible by 18.
4.50. Among 18 successive numbers one is necessarily divisible by 18; let us distinguish this number. The sum of its digits is divisible by 9, hence it is equal to 9 or 18. Indeed, the sum of digits of a three-digit number is equal to 27 only for the number 999, but that number is not divisible by 18, while the sum of digits of a three-digit number cannot be greater than 27. Thus the distinguished number is divisible by 18, while its sum of digits is either 9 or 18, so it is divisible by the sum of its digits.
4.51. The number $100ab\ldots x$ is 37 times greater than the number $ab\ldots x1$. Let us perform the long division of the first number by 37. The fist digit of the quotient is 2, hence $a = 2$. Substituting this value of a in the first number, at the next step we see that $b = 7$. The next digit of the quotient is 1, and there is no remainder. Here we can stop, obtaining 27 as an answer, or continue, obtaining the answers 2 710 027, 271 002 710 027, and so on.
4.52. If a number does not end in 9, then the sum of digits of the next number will be bigger than that of the initial one by 1, so the initial number is not the required one. If the number ends in 9 but not in 99, then the sum of the digits of the next number is less than the sum of its digits by 8, and again this does not meet the requirements of the problem. Now if the number ends in 99 but not in 999, then the sum of digits of the next number will be less than the sum of digits of that number by 17. Hence if the sum of digits of such a number is divisible by 17, then the sum of digits of the next number is also divisible by 17, and this number ends in 00. Obviously, the smallest number with sum of digits 17 that ends in 00, is 8900. Hence the required number is 8899.
4.53. For instance, $13029, 14039$, or 24038. These numbers can be obtained as $13130 - 101$, $14140 - 101$, and $24240 - 202$ respectively.
4.54. For instance 6, 10, 15. More generally the product of two distinct prime numbers p and q has exactly four divisors: 1, p, q and pq.

4.55. The smallest three-digit number divisible by 37 is 111. But it is odd. Hence the smallest even three-digit number divisible by 37 is $111 + 37 = 148$.

4.56. Suppose the greater number is $100a + 10b + c$. Then the smaller number is $100c + 10b + a$, and their difference is $99(a - c)$.

4.57. *First solution.* Fred was able to pay for his breakfast with 11-dollar bills. If we add 33 cups of tea and 11 muffins to his meal, it will still be possible to pay for all of that using 11-dollar bills. This purchase is four times that of Eddie. Since the numbers 4 and 11 have no common divisor, Eddie was able to pay for his meal with 11-dollar bills without change.

Second solution. Let us multiply what Fred bought by 3, obtaining 9 cups of tea, 12 doughnuts, and 15 muffins. Their price is also divisible by 11. Then let us remove 11 doughnuts and 11 muffins (their price is divisible by 11), obtaining Eddie's purchase. So Eddie can pay for his meal in 11-dollar bills.

4.58. After the first deletion, only the digits whose initial positions were even remain, after the second one, only the digits whose initial positions were divisible by 4 remain, after the third, those divisible by 8 and so on. Before the last deletion, the remaining digit will be the one whose initial position number is the largest possible power of 2 that is less or equal to 100, i.e., 64. So that digit is 4.

4.59. The price of each pen changed (increased or decreased) by 5 rubles, hence the total price of all the pens changed by a multiple of 5. But the number 49 is not divisible by 5, so Nick was mistaken.

4.60. See the answers section.

4.61. If the number N is divisible by 24, then it is divisible by 2, 4, and 12. Hence if the last assertion is true, then all the other assertions are true.

4.62. Denote the given numbers by the letters a, b, c, d, e, f, and g, and denote their sum by S. We know that the numbers $S - a$, $S - b$, $S - c$, $S - d$, $S - e$, $S - f$, and $S - g$ are divisible by 5. Hence their sum, which equals $7S - S = 6S$, is also divisible by 5. But then so is S, which means that the numbers $a = S - (S - a)$, …, $g = S - (S - g)$ are also divisible by 5.

4.63. There is an even number of girls sitting in pairs, while the total number of girls is twice that even number, hence the latter is divisible by 4. If it were possible to place the boys in a similar way, their number would also be divisible by 4. By condition, the number of pupils (boys and girls) is 450. This number is not divisible by 4, so no such relocation is possible.

4.64. A number ending in 5 has the form $10n + 5$. Its square is

$$100(n^2 + n) + 25.$$

4.65. The first three digits of the required number are 987. The number 987 is divisible by 7. Hence the last digit is 0 or 7, but the digit 7 is already taken.

4.66. Three cases are possible:

1) four different people went fishing — Nicholas, Peter, and their sons;
2) Peter is the son of Nicholas;
3) Nicholas is the son of Peter.

In the first case, each pair of father and son caught an even number of fish, which is impossible since the total number of fish is 25. In the second case, the number of fish caught by Peter and Nicholas is 6 times that caught by Peter's son, therefore the total number of fish is divisible by 7, which is not the case. Hence only the third case remains: Nicholas is the son of Peter. This is possible since Peter can catch 15 fish, while Nicholas and his son catch 5 each.

4.67. Each of these numbers can be represented as the sum of two numbers each of which is obviously divisible by 11, namely:

$$1001 = 990 + 11 \quad \text{and} \quad 100\,001 = 99\,990 + 11.$$

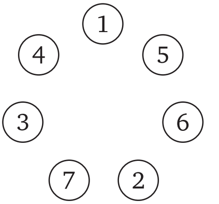
4.68. The number under consideration has the form

$$a + 10b + 100c + 1000c + 10000b + 100000a = \\ = 100001a + 10010b + 1100c.$$

According to Problem 4.67, the numbers 1001 and 100001 are divisible by 11.

4.69. Let us begin with the sequence 1, 2, 4, 8, for which the divisibility is easily verified. Whether the remaining number is divisible by 8 depends only on its last three digits. So one can put 0 in the last position and prepend to it the number 97 536, which is divisible by 4. There are other solutions, e.g., 9, 18, 36, 72, 504.

4.70. For instance, sums of the digits of the numbers 159 999 and 160 000 are divisible by 7.



4.71. a) See the figure. An odd number is not divisible by any even one, hence its neighbours cannot have the same parity. Therefore, odd numbers must stand in pairs. For the pair to the digit 1 we can put 3, 5, or 7. It is easy to consider these three cases and find the answer.

b) As we have already pointed out, odd numbers must stand in pairs. But among the numbers from 1 to 9, there are five odd numbers, and they cannot be organized in pairs.

4.72. a) Let us show that the number written by the teacher is even. Suppose it is odd. Then Daniel answered "No" to the questions on divisibility by 2, 4, 6, and 8, and "Yes" to the questions on divisibility by 3, 5, 7, and 9. But if a number is divisible by 5, 7, and 9, then it is divisible by $5 \cdot 7 \cdot 9 = 315$, so it cannot be a two-digit number. Therefore, one can safely answer "Yes" to the first question.

b) Consider the numbers 18, 40, and 56 and write them in a table (see below). Daniel's answers (+ means "Yes", − means "No") show that the answers to all the questions, except the first one, differ, so that we cannot guarantee answering more than one question correctly. The problem is solved.

	by 2	**by 3**	**by 4**	**by 5**	**by 6**	**by 7**	**by 8**	**by 9**
18	+	+	−	−	+	−	−	+
40	+	−	+	+	−	−	+	−
56	+	−	+	−	−	+	+	−

Now let us explain how it was possible to pick the numbers appearing in the table. Suppose the number written by the teacher is divisible by 8. Then it is also divisible by 2 and by 4. It cannot be divisible by 3, otherwise it would not be divisible by 6, whereas Daniel gave exactly four answers "Yes". Thus the number is not divisible by 3, hence it is not divisible by 9, but is divisible by 5 or by 7 (by exactly one of these two). Therefore, it is either 40, or 80, or 56. Each of these three numbers is divisible by 8 (and by 4) and is not divisible by 9. We can also try to find numbers not divisible by 8 but divisible by 9. Such a number will be divisible by 3, and, being even, it is divisible by 6 as well. Then we have four answers "Yes" already, and we have obtained the number 18 (or 54). It can be shown that the teacher could have written only one of the following eight numbers:

$$12,\ 18,\ 30,\ 40,\ 42,\ 54,\ 56,\ 80.$$

4.73. Searching for the two-digit number, we need not consider:

a) 15, 25, 35, or 45 (it is impossible to assemble a multiple of 5 from the remaining cards);

b) 24 or 42 (it is impossible to assemble an even number from the remaining cards);

c) 12, 32, or 52 (it is impossible to assemble a multiple of 4 from the remaining cards);

d) 54 (it is impossible to assemble a multiple of 9 from the remaining cards).

This decreases the number of variants to be considered.

4.74. Let us say that the number expressed by the digits $9-a$, $9-b$, $9-c$, $9-d$ (in this order) is *dual* to the number expressed by the digits a, b, c, d (in this order). All four-digit numbers without 0's and 9's can be regrouped in pairs of duals; each pair has the sum 9999. Hence the sum of all such numbers is divisible by $9999 = 99 \cdot 101$.

4.75. Suppose that there were k rats. Then each of them ate up $10 : k$ wheels of cheese during the first night. During the second night, each rat ate $5 : k$ wheels of cheese. Hence 7 rats ate $35 : k$ wheels of cheese. This is a whole number, so k divides 35. Moreover, $k > 7$ since not all the rats returned on the second night. The number 35 has only one divisor that is greater than 7, the number 35 itself. Hence $k = 35$, i.e., $35 : k = 1$, and before the invasion of the rats, there were $10 + 1 = 11$ wheels of cheese in the storeroom.

4.76. Put $A = 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \cdot 19 \cdot 21$. Then the numbers

$$\frac{A+1}{2},\ \frac{A+3}{2},\ \frac{A+5}{2},\ \ldots,\ \frac{A+19}{2},\ \frac{A+21}{2}$$

are consecutive natural numbers that are divisible by $1, 3, 5, \ldots, 19$ and 21 respectively.

4.77. The largest remainder under division by 7 is 6, hence the required number is $7 \cdot 5 + 6 = 41$.

4.78. The remainder under division by 7 cannot be more than 6, hence the required numbers have the form $7a + a = 8a$, where $a = 1, 2, \ldots, 6$.

4.79. The identity $100 = 24 \cdot 4 + 4$ shows that 100 hours $=$ $= 4$ days $+ 4$ hours.

4.80. Under division by 3 the numbers 5,10, 15, 20, 25, 30, 35, 40 yield the remainders 2, 1, 0, 2, 1, 0, 2, 1.

4.81. Let us find out how many brownies enter the elevator during the trip from the 1st to the 7th floor and back until the elevator returned to the first floor. At the 1st and 7th floor, one brownie enters, and two enter at each of the other floors. Thus, during one such trip, 12 brownies enter the elevator. Let us divide 1000 by twelve with remainder:$1000 = 83 \cdot 12 + 4$. So, after 83 trips, 4 more brownies will enter the elevator, at the 1st, 2nd, 3rd, and 4th floor.

4.82. a) The remainders under division by 2 of the successive natural numbers are 1, 0, 1, 0, 1, 0, ... Among any four (even any two) successive remainders there is a 0.

b) The remainders under division by 3 of the successive natural numbers are 1, 2, 0, 1, 2, 0, ... Among any four (even any three) successive remainders there is a 0.

c) The remainders under division by 4 of the successive natural numbers are 1, 2, 3, 0, 1, 2, 3, 0, ... Among any four successive remainders there is a 0.

d) None of the numbers 1, 2, 3, 4 is divisible by 5.

4.83. If the given number is divisible by 9, one can append 0 or 9. If the sum of digits of the given number has remainder $n > 0$ under division by 9, one can append the digit $9 - n$.

4.84. If the sum is a multiple of 3 (the smallest possible such number is 9), we can use only 3-ruble bills. If its remainder under division by 3 is 1 (the smallest possible such number is 10), we can take two 5-ruble bills and pay the rest with 3-ruble bills. If the remainder is 2 (the smallest possible such number is 8), we can take one 5-ruble bill and and pay the rest with 3-ruble bills.

4.85. If the quotient equals n, then $m = 13n + 8 = 15n$, so $n = 4$ and $m = 60$.

4.86. The total weight of all the sacks is 139 kg, the weight of sand together with salt is divisible by 3. Hence the weight of flour has remainder 1 under division by 3, i.e., it equals 13 or 31 kg. Accordingly, the salt weighs either 42 kg (and sand, 84 kg) or 36 kg (and sand, 72 kg). The first case is impossible because there cannot be any salt in sacks weighing 31 and 36 kg. In the second case the salt can be only in the sack weighing 36 kg.

4.87. If under division by 9 the number yields the remainder 5, then under division by 3 if yields the remainder 2. If under division by 6 the number yields the remainder 4, then under division by 3 it yields the remainder 1. One and the same number cannot have two different remainders (2 and 1) under division by 3.

4.88. Any odd number can be written as $2k+1$ where k is an integer. The square of an odd number has the form

$$(2k+1)^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1.$$

4.89. The square of an odd number is

$$(2k+1)^2 = 4k(k+1) + 1.$$

One of the two successive numbers k and $k+1$ is even, hence $4k(k+1)$ is divisible by 8.

4.90. The identity $(n-2)(n-1) = n(n-3)+2$ implies that the remainder under division of the product of two successive numbers $n-2$ and $n-1$ by the number n always equals 2.

4.91. The last digit of a square cannot be 3 or 7. The numbers of the form $10a+1$, $10a+5$, and $10a+9$ under division by 4 have the same remainders as $2a+1$. Hence, for odd a, under division by 4, they have the remainder 3, while such numbers cannot be squares (see Problem 4.88).

4.92. Remainders of the division by 3 of successive natural numbers repeat periodically: $1, 2, 0, 1, 2, 0, \ldots$. Hence among six successive natural numbers, two numbers are divisible by 3. The product of those two numbers is divisible by 9.

4.93. Suppose $10a+b$ is divisible by a by b. Then b is divisible by a and $10a$ is divisible by b, hence $b = na$ and $10a = mb$, where m and n are natural numbers. Both numbers a and b are nonzero, hence both sides of the equation $10ab = mnab$ can be divided by ab, obtaining $mn = 10$. For $n = 1$, we obtain the numbers 11, 22, 33, 44, 55, 66, 77, 88, and 99, for $n = 2$, the numbers 12, 24, 36, 48, for $n = 5$, the number 15.

4.94. The last digit of the cube of the number is completely determined by the last digit of the number itself. The last digit of cubes of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are 0, 1, 8, 7, 4, 5, 6, 3, 2, 9, so different last digits of two numbers yield different last digits of their cubes.

4.95. Since the last digit of the cube is 3, the last digit of the number itself is 7. If we replace the star by any digit, we will obtain a number which is greater than $8000 = 20^3$ and less than $27\,000 = 30^3$. Hence the number under consideration is the cube of the number 27. This cube is also divisible by 9, hence the sum of its digits is divisible by 9. Therefore, the star replaced the digit 6.

4.96. Suppose that for the bushes with numbers $1, 3, 5, \dots, 17, 19$ the number of blossoms has remainder 1 under division by 3, while for the others, it is divisible by 3. Then for the bushes with numbers 1 and 19, the total number of blossoms has the remainder 2 under division by 3, whereas for any two neighbouring bushes, the remainder is 1.

4.97. The squares of the numbers 0, 1, and 2 under division by 3 have the remainders 0, 1, and 1. The squares of the numbers a and $a+3$ under division by 3 have equal remainders because $(a+3)^2 - a^2 = 6a+9$ is divisible by 3. Hence the squares of the remainders under division by 3 repeat periodically.

4.98. The squares of the numbers 0, 1, 2, 3, and 4 under division by 5 have remainders 0, 1, 4, 4, and 1. The squares of the numbers a and $a+5$ under division by 5 have the same remainders because $(a+5)^2 - a^2 = 10a + 25$ is divisible by 5. Hence the squares of the remainders under division by 5 repeat periodically.

4.99. If the number $n^2 + n + 1 = (n-2)^2 - 3 + 5n$ is divisible by 5, then the number $(n-2)^2$ yields the remainder 3 under division by 5, but this is impossible (Problem 4.98).

4.100. a) Consider the remainders under division by 5 of the six numbers. There are 5 distinct remainders, hence among these six two are the same. The difference between the numbers having these remainders, is divisible by 5.

b) Suppose, for instance, that all the six numbers have the remainder 1 under division by 5. Then the sum of any two of them has the remainder 2 and is not divisible by 5.

4.101. Basil began with the ninth chocolate from the left, hence of the 992 sweets he ate one out of every seven (the first one in each group of seven). Let us divide 992 by 7 with remainder: $992 = 141 \cdot 7 + 5$. This means Basil ate 142 sweets. After that, 858 sweets remained. Peter began with the seventh chocolate from the left, hence of the 852 chocolates he ate one sweet out of each nine (the first one in each group of seven). Let us divide 852 by 9 with remainder: $852 = 94 \cdot 9 + 6$. Hence Peter ate 95 sweets. Thus 763 chocolates were left.

4.102. The fingers are repeated with period 8, so let us divide 2020 by 8 with remainder: $2020 = 252 \cdot 8 + 4$. The fourth finger (the index finger) will have the number 2020.

4.103. We use the fact that $1967 = 7 \cdot 281$.

First solution. The first condition implies that a has the remainder 5 under division by 7, the second one, that a is odd. Therefore, the remainder under division of a by 14 is 5.

Second solution. We have

$$a = m \cdot 1967 + 68 = (m+1) \cdot 1967 - 1967 + 68 = (m+1) \cdot 1967 - 1899,$$
$$a = n \cdot 1968 + 69 = (n+1) \cdot 1968 - 1968 + 69 = (n+1) \cdot 1968 - 1899.$$

Hence the number $a + 1899$ is divisible both by 1967 and by 1968, so it is divisible by 7 and by 2, i.e., it is divisible by 14. The remainder under division of 1899 by 14 is 9, hence $a + 9$ is divisible by 14, so the remainder under division of a by 14 is 5.

4.104. The remaining numbers are those that have remainders 1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20, or 0 under division by 21. Let us divide 333 by 21 with remainder: $333 = 15 \cdot 21 + 18$. Of the numbers from 1 to $15 \cdot 21$, precisely $15 \cdot 13 = 195$ numbers will remain. Among the 13 remainders listed above, there are 10 that are nonzero and ≤ 18, hence 10 of the 18 numbers exceeding $18 \cdot 21$ remain. In all there remain $195 + 10 = 205$ numbers.

4.105. Suppose the numbers $100a + 10b + c$ and $a + b + c$ are divisible by 7. Then the number

$$100a + 10b + c - 7(14a + b) - 2(a + b + c) = b - c$$

is also divisible by 7. Hence the numbers b and c have the same remainders under division by 7.

4.106. According to Problem 4.105, the last digits of this number have the same remainders under division by 7. Since the sum of all digits equals 7, each of these digits is at most 7 and, moreover, they cannot be the digits 0 and 7, in any order (otherwise the first digit would be 0). Therefore, these digits are the same.

4.107. According to Problem 4.105, the last two digits of the required number have the same remainders under division by 7. Hence these two digits can only be 07, 70, 18, 81, 29, or 92. In the numbers 707 and 770 not all the digits are different. The numbers 18 and 81 have the remainder 4 under division by 7, while the numbers 29 and 92 have the remainder 1. So, among the numbers $100, 200, \ldots, 900$, we are to choose a number that has the remainders $7 - 4 = 3$ or $7 - 1 = 6$ under division by 7. A simple inspection shows that the required numbers are 500 and 300.

4.108. Under division by 101, the numbers 100 101 and 10 010 have the remainders 10 and 11, hence we can put O = 0 and choose numbers B and A so that $\mathrm{B} \cdot 10 + \mathrm{A} \cdot 11 = 101$. We can put B = 9 and A = 1.

4.109. The identity

$$n + (n+1) + (n+2) = 3(n+1)$$

shows that the sum is divisible by 3 and greater than 3.

4.110. a) The number p is odd, hence the numbers $p+1$ and $p-1$ are even.

b) The remainders under the division of natural numbers by 3 are

$$1, 2, 0, 1, 2, 0, \ldots,$$

hence among the three successive numbers $p-1$, p and $p+1$ at least one is divisible by 3. The number p is not divisible by 3.

4.111. a) The remainders in the division of successive natural numbers by 4 are $1, 2, 3, 0, 1, 2, 3, 0, \ldots$. Hence among the four successive numbers $p-1$, p, $p+1$, and $p+2$, at least one is divisible by 4. Both numbers p and $p+2$ are odd, hence one of the numbers $p+1$ or $p-1$ is divisible by 4.

b) For instance, when $p = 13$, none of these numbers is divisible by 5.

4.112. Any number n has the divisors 1 and n. If n has the divisor m, it has the divisor $n : m$ as well. If the total number of divisors is 3, the divisors m and $n : m$ must coincide and the number m must be prime. Hence, n is the square of a prime number.

4.113. Let us underline the numbers 17, 19, 23, and 29, spending 4 rubles. Then underline the number 2, spending one more ruble. After that we will be able to mark all the even numbers (multiples of 2) free of charge. After that, one can underline, free of charge, all the odd numbers not exceeding 15 since any such number n is a divisor of the even number $2n \leq 30$. It remains to underline 21, 25, and 27, and this is also free: 25 is divisible by 5, while 21 and 27 are divisible by 3.

Whatever strategy we adopt, we will have to pay for underlining the prime numbers 17, 19, 23, and 29 since they are not divisors or multiples of any other numbers that are present or may appear on the board. So we will spend 4 rubles only on them. In order to underline anything else, we will have to spend one more ruble at least. Hence, it is impossible to underline all the numbers for less than 5 rubles.

4.114. A prime number $p > 3$ cannot have the remainder 0, 2, 3, or 4 under division by 6 (otherwise it would be divisible by 2 or 3), so only 1 and 5 are possible.

4.115. If $p > 1$ is odd, then the number $5p + 1$ is even and $5p + 1 > 2$, hence it is not prime.

4.116. Both numbers in such a pair cannot be odd (otherwise their sum and difference would be even, while there is only one even prime number). Therefore, among the required prime numbers there is only one even number, namely 2. Hence the difference, the second number and the sum are three consecutive odd numbers. Their differences are not divisible by 3, hence all three have different remainders under division by 3, so one of them is divisible by 3, and then it equals 3 (there are no other prime numbers divisible by 3). Thus, one of the numbers is 2, while their difference or sum equals 3. Hence the required numbers are 2 and 5.

4.117. a) $5 = 2 + 3 = 7 - 2$.

b) The number 2 cannot be represented as the sum of two prime numbers, hence the required prime number is odd. An odd number cannot be represented as the sum or difference of two odd numbers, hence in the sum, as well as in the difference, the prime number 2 must occur.

c) Suppose p is the required prime number. According to item b), the numbers $p - 2$ and $p + 2$ are prime. The numbers $p - 2$, p, and $p + 2$ have different remainders under division by 3, because their pairwise differences are not divisible by 3. There are only three remainders under division by 3, hence one of the three numbers having different remainders under division by 3 is divisible by 3. Among all prime numbers, only 3 is divisible by 3, hence $p - 2 = 3$.

4.118. The required number has the form $1111 \cdot n$, where n is a digit. The number 1111 is the product of two prime factors 11 and 101. If the digit n differs from 1, then the number $1111 \cdot n$ has at least one more prime divisor (besides 11 and 101).

4.119. The difference of the numbers of Smith's flat in the two numerations equals the difference of the number of flats to the left of his entrance and the number of those to the right of it. Therefore, $636 - 242 = 394$ is the number of flats in an even number of entrances. Since $394 = 197 \cdot 2$, while 197 is a prime number, it follows that in one entrance there must be

197 flats. Hence, there are $197 \cdot 5 = 985$ flats in this house. (This answer also shows that the number of flats on different floors must differ.)

4.120. *First solution.* Suppose we divide equally $2 \times 3 = 6$ boxes; 3 extra cookies will remain. But the same six boxes can be divided into two groups of three boxes, and then $2 \cdot 13 = 26$ cookies will remain. Hence $26 - 3 = 23$ cookies can be divided equally between the hikers. Since the number 23 is prime, this is possible only if there are 23 hikers.

Second solution. First let the hikers divide two boxes of cookies, obtaining one extra cookie, then they divide the third box. In that case there should be $13 - 1 = 12$ extra cookies in it, and so in two boxes, there should be $12 \cdot 2 = 24$ extra cookies. But why does there actually remain only one? Because the hikers were able to divide $24 - 1 = 23$ cookies equally. The number is 23 prime, hence there could have been only 23 hikers.

4.121. Suppose the son was born when the father was n years old. When the son's age was k, the father's age was $n + k$. The number $n + k$ is divisible by k if and only if n is divisible by k. Hence we must find a number n that has exactly 8 divisors and such that $2n \leq 75$, i.e., $n \leq 37$. The numbers that have 8 divisors can be represented in one of the following forms: p^7, p^3q, pqr (where p, q, and r are distinct prime numbers). It is easily verified that only the numbers $24 = 2^3 \cdot 3$ and $30 = 2 \cdot 3 \cdot 5$ have 8 divisors and are no greater than 37.

4.122. Among the first 9 prime numbers, the only even one is 2. The sum of the numbers in the row containing 2 is even since sum of two odd numbers and one even number is even and the sum of the numbers in the rows not containing 2 is odd. Hence the sums of numbers in some two rows are different and it is impossible to obtain a magic square.

4.123. Since $1000 = 5^3 \cdot 2^3$, it follows that each of the numbers may contain only 2's and 5's in its prime factorisation. Both 2 and 5 cannot occur in the factorisation of the same number, because otherwise it will be divisible by 10. Therefore, one of the numbers is 5^3, the other is 2^3, and their sum is $5^3 + 2^3 = 125 + 8 = 133$.

4.124. If the number p is divisible by 3 and greater than 3, then it is not prime. Suppose p is divisible by 3 and greater than 1. Then p^2 has the remainder 1 under division by 3 (Problem 4.97), hence $p^2 + 2$ is divisible by 3. Since $p^2 + 2 > 3$, this number is composite. Hence p can only be 3. This number yields the answer, because $3^2 + 2 = 11$ is prime.

4.125. If the numbers p and p^2+2 are prime, then $p = 3$ (Problem 4.124), so that $p^3 + 2 = 29$ is prime.

4.126. If p is not divisible by 3, then the number p^2 has the remainder 1 under division by 3 (Problem 4.97), hence the number $2p^2$ has the remainder 2 under division by3, while the number $2p^2 + 1$ is divisible by 3.

4.127. If p is not divisible by 5, then under division by 5 the number p^2 can have remainder 1 or 4 (Problem 4.98), hence either $p^2 + 4$ or $p^2 + 6$ is divisible by 5. For $p = 5$ the numbers p, $p^2 + 4 = 29$, and $p^2 + 6 = 31$ are prime.

4.128. *First solution.* If p is a prime number and $p > 3$, then p is odd and is not divisible by 3. The square of the odd number p has the remainder 1 under division by 8 (Problem 4.89), hence $p^2 - 1$ is divisible by 8. The square of the number p which is not divisible by 3 has the remainder 1 under division by 3 (Problem 4.97), hence $p^2 - 1$ is divisible by 3. Therefore, $p^2 - 1$ is divisible by $8 \cdot 3 = 24$.

Second solution. Represent the number $p^2 - 1$ as the product of the factors $p - 1$ and $p + 1$. One of two consecutive even numbers $p - 1$ and $p + 1$ is divisible by 4. One of three consecutive numbers $p - 1$, p, and $p+1$ is divisible by 3, and it cannot be the number p. Hence the number $(p - 1)(p + 1)$ is divisible by $2 \cdot 4 \cdot 3 = 24$.

4.129. Represent the number $p^2 - q^2$ as the product of the factors $p - q$ and $p + q$. The numbers $p - q$ and $p + q$ are even and their difference $2q$ is not divisible by 4. Hence one of these numbers is divisible by 4. The number $2q$ is not divisible by 3, so the numbers $p - q$ and $p + q$ have different remainders under division by 3. If none of these numbers is divisible by 3, then their sum would be divisible by 3. But the number $(p-q)+(p+q) = 2p$ is not divisible by 3. Hence $(p-q)(p+q)$ is divisible by $2 \cdot 4 \cdot 3 = 24$.

4.130. For $n = 41$ the number n^2+n+41 is divisible by 41 and is greater than 41. Note that for n from 1 to 39 the number $n^2 + n + 41$ *is* prime.

4.131. Let us prove that any prime number $p > 11$ can be represented as the sum of two composite numbers. Since any such number is odd, it follows that the number $p - 9 > 2$ is even and, therefore, composite. Hence $p = (p - 9) + 9$ is the required representation. On the other hand, it can be verified directly that the numbers 2, 3, 5, 7, and 11 cannot be represented as the sum of two composite numbers.

4.132. The age of of the youngest child cannot be even, because otherwise the ages of the elder children will not be prime. The age of the youngest cannot end in 1, 3, 7, or 9, since otherwise the age of one of the elder children will be divisible by 5. The only prime number satisfying these conditions is 5. A verification shows that if the age of the younger child is 5, then the ages of all the elder children are prime numbers.

4.133. Suppose there were n people in the group, and each sent k letters. Then each one sent $k(n-1)$ letters, and a total of $k(n-1)n$ letters were sent. Hence the number 440 is the product of three factors, two of which differ by 1. Since $440 = 2^3 \cdot 5 \cdot 11$, it follows that we can carry out an exhaustive search with the following restrictions:

1) $n < 22$ (since $22 \cdot 21 > 440$);

2) the numbers n and $n-1$ have no prime factors other than 2, 5, and 11.

As the result, we obtain three variants:

$$440 = 4 \cdot 10 \cdot 11 = 22 \cdot 4 \cdot 5 = 220 \cdot 1 \cdot 2.$$

4.134. c) Each divisor has the form $p^k q^l$. The number of choices of the exponent k is $m+1$ (it can be any whole number from 0 to m), while the number of choices of the exponent l is $n+1$.

4.135. Note that $234 = 2 \cdot 3 \cdot 3 \cdot 13$. The largest factor obtained by Lena could be equal to $9 + 4 = 13$. There was such a factor in the product, because smaller numbers are not divisible by 13, while 234 is. Hence, the last digit of the written number is 9. The product of the other three factors is 18. The digit 1 is not one of those three, because the first digit cannot be zero. The only remaining possibility is $18 = 2 \cdot 3 \cdot 3$. The third digit is in this case 0, while for the two first digits there are two possibilities.

4.136. a) *First solution.* Consider the product of all prime numbers from 2 to 11. It equals $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$. Now consider the numbers from 2312 to 2321. Among these ten numbers there are no primes. Indeed, the even numbers are divisible by 2, the numbers 2313 and 2319 are divisible by 3, 2315 is divisible by 5, 2317, by 7, and 2321, by11.

Second solution. Put $n = 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 11$. Then the numbers $n+2$, $n+3$, …, $n+11$ are composite: they are divisible by $2, 3, \ldots, 11$ respectively.

b) In the interval from 30 to 39, there are two prime numbers (31 and 37).

c) In the interval from 22 to 31, there are three prime numbers (23, 29, and 31).

d) In the interval from 3 to 12, there are four prime numbers (3, 5, 7, and 11).

e) All the numbers from 2325 to 2332 are composite: 2327 is divisible by 13, 2329 is divisible by 17, 2331 is divisible by 3. One can verify that the number 2333 is prime. Hence in the interval from 2325 to 2334 there is exactly one prime number.

f) In the interval from 2 to 11, there are five prime numbers (2,3, 5, 7, and 11). Among ten consecutive natural numbers, there must be 5 even ones, and among the odd ones, only one can be divisible by 5. Hence among the ten consecutive numbers greater than 2, there can be no more than five primes, while among those greater than 5, no more than four. Among the numbers from 3 to 12, from 4 to 13 and from 5 to 14 there are four prime numbers. So there are five prime numbers only in the interval from 2 to 11.

4.137. Suppose that there is a finite quantity of prime numbers. Consider the product of all the prime numbers and add 1 to it. On the one hand, the obtained number has no common divisors with any prime number (under division by any one of them it yields the remainder 1). On the other hand, it is not part of the list of all prime numbers, hence it must have a prime divisor.

4.138. One has $c = a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, where a, b, and c are primes. Since number c is prime, one has $a - b = 1$. The difference between two prime numbers is equal to 1 only in the case of the numbers 2 and 3. Then $c = 19$.

4.139. Let us decompose the number 203 into prime factors: $203 = 7{\cdot}29$. All the other factors must be ones. The sum of all these factors must be equal to 203, hence in the product there must be $203 - (7 + 29) = 167$ ones.

4.140. During one month the pupils collected $49\,685 : 5 = 9937$ rubles. This sum is the product of the number of classmates by the contribution of each. The number 9937 may be represented as product of two factors in only two ways: $9937 = 9937 \cdot 1 = 19 \cdot 523$. But the number of pupils in a class cannot be 9937, nor 523. Therefore, there is only one possibility: 19 students contributed 523 rubles each every month.

4.141. Let us decompose 420 into factors:

$$420 = 6 \cdot 7 \cdot 10 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7.$$

Now it is easy to regroup these factors and the factor 1 into five groups so as to obtain the sum 20: $7 + 5 + 3 + 4 + 1 = 20$.

4.142. Note that $\mathrm{B} = 1$. Indeed, if $\mathrm{B} \geq 2$, then

$$\mathrm{BAO} \times \mathrm{BA} \times \mathrm{B} \geq 200 \cdot 20 \cdot 2 = 8000 > 2002.$$

Decompose 2002 into factors: $2002 = 2 \cdot 7 \cdot 11 \cdot 13$. The number BA is a two-digit divisor of the number 2002 beginning with the digit 1, hence BA can only be 11, 13, or $2 \cdot 7 = 14$. Since $\mathrm{B} \neq \mathrm{A}$, it follows that $\mathrm{BA} \neq 11$. If $\mathrm{BA} = 13$, then $\mathrm{BAO} = 2002 : 13 = 154$, hence the digit A is equal to both 3 and 5, which is impossible. The only remaining variant is $\mathrm{BA} = 14$. In this case,

$$\mathrm{BAO} = 2002 : 14 = 143$$

is the answer.

4.143. It follows from the factorisation $2001 = 3 \cdot 23 \cdot 29$ that the number 2001 can be represented as the product of two two-digit numbers only in the following ways: $69 \cdot 29$ or $23 \cdot 87$. Only the first version works.

4.144. Denote by p the number of entrances in the house, by f, the number of floors, and by k the number of flats on each floor. Then $p \cdot f \cdot k = 105 = 3 \cdot 5 \cdot 7$, and the numbers 3, 5, and 7 are prime. Having in mind that $1 < p < k < f$, we obtain $f = 7$.

4.145. Denote by s the number of watchmen in each team, by b, the number of teams, and by n, the number of nights during which each watchman slept. Then $s \cdot b \cdot n = 1001$. But $1001 = 7 \cdot 11 \cdot 13$, and the numbers 7, 11, and 13 are prime. Having in mind that $s < n < b$, we obtain $s = 7$.

4.146. The required number should have the form $2^m 3^n$, where m is an odd number divisible by 3, and n is a even number that has the remainder 2 under division by 3. The smallest such number is $2^3 \cdot 3^2 = 72$.

4.147. Decompose the number 2007 into prime factors: $2007 = 3 \cdot 3 \cdot 223$. In the case of such a decomposition, Ivan's grades are two twos and three threes. Other possible factorisations of the number 2007 are $2007 = 9 \cdot 223 = 3 \cdot 669$. But since the grade 9 does not exist, these factorisations of 2007 could not have come from Ivan's grades. Since Ivan has more threes than twos, he will get the final grade three for the trimester.

4.148. The required number is a divisor of the number 2008. Decompose the number 2008 into prime factors: $2008 = 2 \cdot 2 \cdot 2 \cdot 251$. Let us write out all the divisors of the number 2008: 1, 2, 4, 8, 251, 502, 1004, 2008. Finding the sum of digits for each of them, we can verify that the conditions of the problem are satisfied only for the number 251 ($2008 = 251 \cdot (2 + 5 + 1)$).

4.149. a) Let us divide the chests into three pairs. If we open two chests from the same pair, then the coins in these chests can be redistributed so that the number of coins in the two chests will be the same. Hence the total number of coins in each pair of chests is even and so the total number of coins in the six chests is also even. Now let us divide the chests into two triples. The number of coins in each triple is divisible by 3, hence so is the total number of coins in all the chests. Thus, the total number of coins is divisible by 2 and 3, and so by 6 as well. Therefore, all the coins can be distributed equally in the six chests.

b) Let us divide the chests into 11 groups with 7 chests in each. If we open 7 chests from one of the groups, then the coins in them can be redistributed so that the number of coins in all the chests will be the same. Hence the total number of coins in each group of chests is divisible by 7, and so the total number of coins in the 77 chests is divisible by 7. Let us divide the chests in 7 groups of 11 chests each. Now the number of coins in each group is divisible by 11, and so the total number of coins is divisible by 11. Thus, the total number of coins is divisible by the prime numbers 7 and 11, and so it is divisible by their product 77. Therefore, all the coins can be distributed equally in the 77 chests.

4.150. Use the fact that $\mathrm{LCM}(12, 15) = 60$.

4.151. Let us enumerate in increasing order all the days after March 31: the first day is April 1st, ..., the 31st day is May 1st, and so on. Nick went to the movies on the days whose numbers are divisible by 4, Serge went on the days whose numbers are divisible by 5, Ian, on the days whose numbers are divisible by 6. They will be at the movies together on days whose numbers are divisible by $\mathrm{LCM}(4, 5, 6) = 60$. The first such day is May 30.

4.152. The sum of digits of the second number is 9 (there was no carry when adding 1 to the first number since the first number does not contain the digit 9). Hence the second number is divisible by 9 and by 8. The smallest such number is $\mathrm{LCM}(8, 9) = 72$. The previous number, 71, has the sum of digits equal to 8.

4.153. We are to find a natural number which is smaller than 50 and is divisible by $\mathrm{LCM}(2, 3, 7) = 42$. There is only one such number, hence there are 42 pupils in the class; 6 of them got the best grade (A); 14 got B's; 21, C's. Therefore, exactly 1 pupil got the unsatisfactory grade.

4.154. Both numbers are divisible by 111, and

$$111\,111 : 111 = 1001 \quad \text{and} \quad 111\,111\,111 = 1\,001\,001.$$

The numbers 1001 and 1 001 001 have no common divisors. Indeed, any of their common divisors must also divide their difference

$$1\,001\,001 - 1001 = 1\,000\,000.$$

4.155. If we add 1 to the required number, the resulting number will be divisible by $2, 3, \ldots, 8$. Hence the required number is

$$\mathrm{LCM}(2, 3, 4, \ldots, 8) - 1 = 8 \cdot 3 \cdot 5 \cdot 7 - 1 = 840 - 1 = 839.$$

4.156. If we add 1 to the required number, then the resulting number will be divisible by $7 \cdot 8 \cdot 9 = 504$. The required number is a three-digit number, hence it can only be $504 - 1 = 503$, since the number $2 \cdot 504 - 1 = 1007$ has four digits.

4.157. The number 121 212 121 212 is divisible by 121 212 (the quotient is 1 000 001). If a number n is divisible by m, then

$$\mathrm{LCM}(n, m) = n \quad \text{and} \quad \mathrm{GCD}(n, m) = m.$$

4.158. The difference between two numbers is divisible by any of their common divisors. Hence any common divisor of the given numbers is a divisor of the number 8. But the number 8 has no odd divisors other than 1.

4.159. The numbers n and $n + 1$ are coprime, hence both must be squares. But two consecutive natural numbers cannot be both squares.

4.160. The required number is a divisor of the number

$$12\,354 - 12\,345 = 9,$$

hence it equals 1, 3, or 9. The sum of digits of each of these numbers is 15, hence they are all divisible by 3 and are not divisible by 9.

4.161. The required number is a divisor of the number

$$123\,456\,798 - 123\,456\,789 = 9,$$

hence it is 1, 3 or 9. The sum of digits of each of the given numbers equals 45, hence they all are divisible by 9.

4.162. Denote the required number by n. Then $100 = an + 4$ and $90 = bn + 18$ for some integers a and b, hence $an = 96$ and $bn = 72$. The number n is a divisor of the number $\mathrm{GCD}(96, 72) = 24$ and is greater than 18, hence it equals 24

4.163. Suppose the GCD of three distinct natural numbers equals d. Then these numbers are ad, bd, and cd, where a, b, and c are distinct natural numbers. One of the three distinct natural numbers a, b and c is at least 3, hence $3d \leq 2000$. Therefore, $d \leq 666$.

4.164. The required numbers have the form $36a$ and $36b$, where a and b are coprime. By condition $36a + 36b = 288$, i.e., $a + b = 8$. Coprime numbers whose sum equals 8 are 1 and 7 or 3 and 5.

4.165. If the required numbers are equal to $18a$ and $18b$, then

$$\mathrm{LCM}(a, b) = \frac{630}{18} = 35$$

and the numbers a and b are not divisible by each other. Hence the numbers a and b are equal to 5 and 7.

4.166. For instance, the numbers from 13 to 18. Indeed,

$$\mathrm{LCM}(13, 14, 15) = 13 \cdot 14 \cdot 15 = 2730$$

and

$$\mathrm{LCM}(16, 17, 18) = 16 \cdot 17 \cdot 9 = 2448.$$

4.167. Two numbers have the same remainders under division by the number a if and only if their difference is divisible by a. Hence if two numbers have the same remainders under division by the numbers a, b, c, $\ldots$, then the difference between these numbers is divisible by a, b, c, $\ldots$, and so this difference is divisible by the LCM of the numbers a, b, c, $\ldots$, hence the numbers have the same remainders under division by the LCM of the numbers a, b, c, $\ldots$

4.168. If there is an even number of coins in one chest and an odd number of them in another, then it is impossible to divide the coins between them equally. Hence the number of coins in all the chests have the same parity.

The total amount of coins in the first three chests is divisible by 3. If we replace the third chest by the fourth one, then the divisibility by 3 will be preserved. This means that the number of coins in the fourth chest has the same remainder under division by 3 as that in the third.

In the same way, we can prove that in any two chests, the number of coins has the same remainder under division by 3. Hence the remainders under division by 3 are the same for all the chests.

Continuing this argument, we can prove that the number of coins in each chest has the same remainder under division by 2, 3, 4, 5, 6, and 7. Hence, these numbers have the same remainders under division by $\mathrm{LCM}(2, 3, 4, 5, 6, 7) = 420$. But 420 is not divisible by 8, hence these numbers can have distinct remainders under division by 8, which hampers the equal division of the coins between the eight chests.

Indeed, there could be 421 coins in one of the chests, and one coin in each of the seven others. Then in any two chests there are either 2 or 422 coins, both numbers are even. In any three chests there are either 3 or 423 coins, each of these numbers being divisible by 3, and so on. However, the total number of coins is 428, which is not divisible by 8. Hence in this case it is impossible to equally divide the coins between the eight chests.

On the other hand, there could have been an equal number of coins in all the chests from the start. Therefore, we cannot answer the question if we don't know how many coins there are in the chests.

Chapter 5. Fractions

5.1. The cyclist walked twice as less distance than rode, and this took him twice as much time.

5.2. Tony's speed is $\frac{9}{10}$ of Serge's speed, hence at the moment when Alex finished, Tony covered $\frac{9}{10}$ of the distance covered by Serge, i.e., $90 \cdot \frac{9}{10} = 81$ m.

5.3. Volumes of the second and third pails constitute $\frac{3}{2}$ and $\frac{4}{3}$ of that of the first, hence the volume of the first pail in litres is divisible by 6. If the pails contain 6, 9, and 8 litres, then together they contain 23 litres. If they contain 12 litres, 18, and 16 litres, then together they contain 46 litres, which is impossible.

5.4. The first condition means that the book costs more than

$$\frac{11}{10} \text{ dollars} = 1 \text{ dollar } 10 \text{ cents.}$$

The second condition means that the book costs less than

$$\frac{10}{9} \text{ dollars} = 1 \text{ dollar } 11\frac{1}{9} \text{ cents.}$$

Hence the book costs 1 dollar 11 cents.

5.5. *First solution.* The white baby octopuses constitute one third of all the children and they did not change colour. If we add 10 and 18, we obtain the total number of children to which the number of white children has been added, i.e., $\frac{4}{3}$ of the number of all children. Hence, $\frac{4}{3}$ of the children in the family is 28, hence there were 21 children.

Second solution. After changing colour, the number of green children has become greater by $18 - 10 = 8$ than that of the blue ones. Hence, $8 : 2 = 4$ blue octopuses have turned green. Thus, before the children changed colour, the total number of white and green baby octopuses was $18 - 4 = 14$, so there were $14 : 2 = 7$ of each colour. Hence, the total number of children is $3 \cdot 7 = 21$.

5.6. The level of water with one pump turned on rose by 2 cm a minute, and with two pumps on, it lowered by 2 cm a minute. Hence one pump pumps out 4 cm/min, so two pumps pump out 8 cm/min. Thus, it will take $10 : 8 = \frac{5}{4}$ minutes to remove the remaining water.

5.7. Two thirds of the girls saw with the right eye what was too early to see, while two thirds of the boys saw it with the left eye. So two thirds of the pupils, i.e., 22 people, saw what they were not supposed to see.

5.8. a) The equation $\frac{n+m}{2m} = 2\frac{n}{m}$ implies that $m = 3n$;

b) the equation $\frac{n+m}{2m} = 3\frac{n}{m}$ implies $m = 5n$;

c) the equation $\frac{n+m}{2m} = 4\frac{n}{m}$ implies $m = 7n$.

5.9. The product of the numerator of the required fraction by 11 must be greater than $5 \cdot 15 = 75$ and smaller than $6 \cdot 16 = 90$. The inequalities

$$66 < 75 < 77 < 88 < 90 < 99$$

show that the numerator can be only 7 and 8.

5.10. See the answers section.

5.11. These fractions are equal to $1 - \frac{1}{5}$, $1 - \frac{1}{4}$, and $1 - \frac{1}{6}$.

5.12. First let us compare the fractions $1 - \frac{78}{79} = \frac{1}{79}$ and $1 - \frac{90}{91} = \frac{1}{91}$. Clearly, $\frac{1}{79} > \frac{1}{91}$, hence $\frac{78}{79} < \frac{90}{91}$.

5.13. The equation $\frac{m}{n} = \frac{m+4}{n+10}$ implies $10m = 4n$, so $5m = 2n$.

5.14. The sum of the numerator and denominator will not change if we subtract some number from one of them and add the same number to the other. This sum equals 1000, hence the resulting fraction before its reduction was $\frac{100}{900}$. In order to obtain it, one should subtract and add the number 437.

5.15. For instance, $m = 2$ and $n = 7$.

5.16. The numbers n and $n+1$ are coprime, the numbers $n+2$ and $n+1$ are also coprime. Hence the numerator and denominator cannot have any common divisors.

5.17. Suppose that Ivan Knownothing cannot succeed in arranging the numbers as required.

First solution. Let us join each pair of neighbouring numbers by clockwise oriented arrows. On each arrow let us write the result of division of the number at its tail by the number at its head. By hypothesis, this will be p or $\frac{1}{p}$, where p is a prime number. Each of the written numbers played the role of the dividend exactly once and the role of the divisor exactly once. Hence the product of all numbers written on the arrows is 1. Therefore, for each number of the form p, there is, among the numbers written on the arrows, a corresponding number of the form $\frac{1}{p}$. So there is an equal number of whole numbers and proper fractions, which is impossible since their total number is 2021.

Second solution. Consider the decomposition of each of the numbers placed by Ivan into prime factors. Let us count the quantity of prime factors in each of the numbers (if there are repetitions, each prime is counted the number of times it occurs: for example, $8 = 2^3$ has 3 prime factors; the number 1 has zero prime factors). The ratio of neighbouring numbers is a prime number, hence the quantities of prime factors in the factorisations of these numbers have different parity, hence the quantities of prime factors of the first and the 2021th number have the same parity; on the other hand, these numbers stand next to each other, so their quantities of prime factors must have different parity. We arrived at a contradiction.

5.18. Pinocchio slept during $\frac{2}{3}$ of half of his route, i.e., $\frac{1}{3}$ of his entire route. So he was awake $\frac{2}{3}$ of the route.

5.19. The sum of all these fractions is greater than 1, since $\frac{1}{5} + \frac{1}{6} > \frac{1}{3}$.

5.20. On the first day Basil read $\frac{1}{2}$ and on the second day, $\frac{1}{3}$ of $1 - \frac{1}{2}$, i.e., $\frac{1}{6}$. Hence in 2 days Basil read $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ of the book, and on the third day, he read the remaining $\frac{1}{3}$ of the book. Thus he succeeded in reading the whole book in 3 days.

5.21. From the first flag to the eighth there are 7 intervals between flags, from the first to the twelfth there are 11 intervals. The athlete covers each interval in $\frac{8}{7}$ s, hence he will cover 11 intervals in

$$11 \cdot \frac{8}{7} = \frac{88}{7} = 12\frac{4}{7} \text{ s.}$$

5.22. The speed of the train with respect to the passenger is

$$45 + 36 = 81 \text{ km/h}, \quad \text{i.e.,} \quad \frac{81\,000}{60 \cdot 60} = \frac{45}{2} \text{ m/s.}$$

Hence the length of the first train is

$$\frac{45}{2} \cdot 6 = 135 \text{ m.}$$

5.23. In the first and second truck, let us place 3 full barrels, 1 half filled barrel, and 3 empty barrels, and on the third one, 1 full barrel, 5 half full barrels, and 1 empty one. Then there will be 7 barrels in each truck. Since $3 + \frac{1}{2} = 1 + \frac{5}{2}$, the amount of water will be the same in the three trucks.

5.24. The sum $1\frac{1}{2} + 2\frac{1}{2} = 4$ kg contains two increments induced by the shift of the pointer, while the $3\frac{1}{2}$ kg reading of the scales contains only one such increment. Hence the shift equals

$$4 - 3\frac{1}{2} = \frac{1}{2} \text{ kg.}$$

Thus, the correct weight of each bunch of bananas is half a kilo less than that shown by the balance.

5.25. One third of the tank was poured into each of the three cans. Hence, the volume of the first can equals $2 \cdot \frac{1}{3} = \frac{2}{3}$ of the tank, the volume of the second, $\frac{3}{2} \cdot \frac{1}{3} = \frac{1}{2}$ of the tank, the third, $\frac{4}{3} \cdot \frac{1}{3} = \frac{4}{9}$ of the tank.

All these quantities are whole numbers, hence the volume of the tank is divisible by $\mathrm{LCM}(2, 3, 9) = 18$. Therefore, the required minimal capacity of the tank is 18 l.

5.26. Subdivide the square into a 500×500 square grid so that the side of each small square should be $\frac{1}{500}$ cm and the total number of small squares should be 500^2. The sum of the lengths of the sides of small squares is equal to

$$4 \cdot 500^2 \cdot \frac{1}{500} = 2000 \text{ cm.}$$

5.27. If we cut off one fourth of the string (to do that, we fold it in half twice), then 50 cm will remain. Indeed, $\frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$.

5.28. Molly brewed 57 cups, hence she had at most $\frac{57}{2} = 28\frac{1}{2}$ teabags. Tina brewed 83 cups, hence she had at least $\frac{83}{3} = 27\frac{2}{3}$ teabags. They had the same number of teabags, and the only whole number which is no greater than $28\frac{1}{2}$ and no less than $27\frac{2}{3}$ is 28.

5.29. Six diggers in 2 hours will dig $2 \cdot 3 = 6$ holes, hence in 5 hours they will dig $\frac{5}{2} \cdot 6 = 15$ holes.

5.30. Six diggers in 3 hours will dig $2 \cdot 3 = 6$ holes, hence in 5 hours they will dig $\frac{5}{3} \cdot 6 = 10$ holes.

5.31. Adding one third of Pooh's share to Piglet's share, we have tripled the latter. Hence one third of Pooh's share was twice Piglet's share, and so Pooh's entire portion was 6 times greater than Piglet's. Hence, initially Pooh had $\frac{6}{7}$ of the pie, while Piglet had $\frac{1}{7}$.

5.32. In one hour father and son together paint $\frac{1}{12}$ part of the fence, and father alone paints $\frac{1}{21}$ of it. Hence in one hour the son by himself paints $\frac{1}{12} - \frac{1}{21} = \frac{1}{28}$ of the fence.

5.33. After the first day, the hiker still had to cover $\frac{2}{3}$ of the route, after the second, $\frac{2}{3}$ of the remaining route, i.e., $\frac{2}{3} \cdot \frac{2}{3}$ of the entire route, after the third, $\frac{2}{3}$ of the new remainder, i.e., $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$ of the whole route. Thus, $\frac{8}{27}$ of the whole route constitute 32 km, hence the whole route is $\frac{27}{8} \cdot 32 = 108$ km.

5.34. The first boy contributed one part, the second, n parts. Together they contributed $n+1$ equal parts, hence the first boy contributed $\frac{1}{n+1}$ of the whole sum.

5.35. The first contributed $\frac{1}{3}$ of the whole sum, the second, $\frac{1}{4}$ of the whole sum, the third, $\frac{1}{5}$ of the whole sum, while the fourth paid 1300. Together, the first three paid $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$, while the fourth paid $1 - \frac{47}{60} = \frac{13}{60}$. Therefore, the boat cost

$$\frac{60}{13} \cdot 1300 = 6000 \text{ rubles.}$$

5.36. The interval between trains should be multiplied by $\frac{4}{5}$, hence the number of trains should be multiplied by $\frac{5}{4}$. As the result, we obtain 30 trains, hence the number of trains should be increased by 6.

5.37. Let us calculate successively:

$$1-\frac{1}{2}=\frac{1}{2},\qquad \frac{1}{2}-\frac{1}{4}=\frac{1}{4},\qquad \frac{1}{4}-\frac{1}{8}=\frac{1}{8},$$
$$\frac{1}{8}-\frac{1}{16}=\frac{1}{16},\quad \frac{1}{16}-\frac{1}{32}=\frac{1}{32},\quad \frac{1}{32}-\frac{1}{64}=\frac{1}{64}.$$

5.38. The area of the big field that was mowed in the first half of the day is twice the area of the same field that was mowed in the second half of the day. Hence in half a day the team can mow $\frac{2}{3}$ of the bigger field. Of the smaller field, there remained, after the first day, the part equal to $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$ of the bigger field. During one day, the whole team can mow $\frac{4}{3}$ of the bigger field, hence the number of mowers is $\frac{4}{3}:\frac{1}{6}=8$.

5.39. Piglet and Rabbit together ate half of what was eaten by Winnie the Pooh, and half of what he did not eat. What Pooh ate and what he did not eat is all the honey in the barrel. Hence Piglet and Rabbit ate half the honey. The other half was eaten by Pooh and Eeyore. Eeyore ate one tenth of the barrel, hence Pooh ate $\frac{1}{2}-\frac{1}{10}=\frac{2}{5}$ of the barrel. So, Pooh did not eat $\frac{3}{5}$ of the barrel, while Rabbit ate half of that amount, i.e., $\frac{3}{10}$.

5.40. First we can divide three apples between the boys (each will get one fourth of an apple), and then divide between them the remaining four apples (one third for each). Such a division of the apples corresponds to the identity $\frac{7}{12}=\frac{1}{4}+\frac{1}{3}$.

5.41. Let us replace each of the covers of the two books by ten leaves of paper. Then the number of leaves in the two books will be $(500/2)\cdot 2+10\cdot 4=540$. Now two different arguments are possible.

First solution. The distance between the bookmarks equals the thickness of $540:3=180$ leaves. In the first book, there are $540:4=135$ leaves after the bookmark. In the second book, before the bookmark there are $180-135=45$ leaves, of which 10 leaves constitute one cover. Hence, the bookmark lies after the leaf number $250-(45-10)=215$, i.e., after page 430.

Second solution. The thickness of each cover constitutes $\frac{10}{540}=\frac{1}{54}$ of the thickness of the two books. Between the bookmarks there is one third of the width of two books. This one third contains one fourth of this width (half the first book) and $\frac{1}{54}$ of this width (one of the covers of the second book). Thus, the number of leaves in the second book from the back cover to the bookmark is $\frac{1}{3} - \frac{1}{4} - \frac{1}{54} = \frac{7}{108}$ of the total sum of the thicknesses of the two books. So, the number of leaves in the second book is $540 \cdot \frac{7}{108} = 35$; further we argue as in the first solution.

5.42. We have $\frac{1}{x} = \frac{2}{73} - \frac{1}{60} - \frac{1}{219} - \frac{1}{292} = \frac{1}{365}$. Hence, $x = 365$.

5.43. a) Exactly one fraction with denominator 2 is written, exactly two with denominator 3, exactly three with denominator 4, and so on. Hence up to the fraction $\frac{1}{65}$ not included, the number of fractions is

$$1 + 2 + 3 + \ldots + 62 + 63 = \frac{64 \cdot 63}{2} = 2016.$$

After these 2016 fractions, four more fractions are written, so the total number of written fractions is $2016 + 4 = 2020$.

b) If 2016 fractions have been written, then among them all the fractions with denominators $2, 3, \ldots, 64$ are present. Consider the fractions with the same denominator, written in increasing order. If the denominator is odd, then each of the fractions of the first half is less than $\frac{1}{2}$, while each of the fractions of the second half is greater than $\frac{1}{2}$. If the denominator is even, then one of the written fractions is equal to $\frac{1}{2}$, while among the remaining fractions half are less than $\frac{1}{2}$. Thus, the difference between the number of blue fractions and the number of red fractions equals the number of fractions equal to $\frac{1}{2}$. These are the fractions $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, …, $\frac{32}{64}$, and there are 32 of them.

5.44. If $\frac{a+10}{b+10} = 2 \cdot \frac{a}{b}$, then

$$b(a + 10) = 2a(b + 10),$$

so $a(b + 20) = 10b$. By hypothesis, b and a have no common divisors, hence a is a divisor of 10. From the given numerator a, we find the denominator b by the formula $b = \frac{20a}{10-a}$. Clearly, $a \neq 10$ (for $a = 10$ we would have to divide by zero). For $a = 1$ the value of b is not whole; for $a = 2$ we obtain $b = 5$, while for $a = 5$, we obtain $b = 20$. The last case will not do, because the fraction $\frac{5}{20}$ is reducible.

5.45. a) The simplest way is to choose three fractions with numerators equal to 1. For such fractions, the second condition always holds, hence we only need that the sum should be a whole number. The identity $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$ hints what fractions to take.

b) Let us multiply the fractions from item a) by a whole number n. The sum of the fractions $\frac{n}{2}$, $\frac{n}{3}$, and $\frac{n}{6}$ is equal to the whole number n. The sum of the reciprocal fractions $\frac{2}{n}$, $\frac{3}{n}$, and $\frac{6}{n}$ is equal to $\frac{2+3+6}{n} = \frac{11}{n}$. For $n = 11$, this sum is a whole number, and this triple of reciprocal fractions constitutes an answer.

5.46. Of the possible examples, let us indicate two:

$$\frac{7}{4}+\frac{6}{8}+\frac{5}{1}+\frac{3}{2}=9, \quad \frac{5}{4}+\frac{6}{8}+\frac{9}{3}+\frac{2}{1}=7.$$

5.47. The sum of the numerator and denominator equals 101. Hence the greater the numerator of the fraction, the smaller its denominator, and so the greater the fraction itself. The fraction $\frac{25}{76}$ is still less than $\frac{1}{3}$, while $\frac{26}{75}$ is greater than than $\frac{1}{3}$. Alternatively, to find the greatest n for which the fraction $\frac{n}{101-n}$ is still less than $\frac{1}{3}$ one can proceed as follows. Suppose $\frac{n}{101-n} < \frac{1}{3}$ and $101 - n > 0$. Then $4n < 101$. The greatest n satisfying this inequality is 25.

5.48. The numbers a, b, c are positive and less or equal to 9, hence

$$9+\frac{1}{c} \geq b+\frac{1}{c}, \quad a+\frac{2020}{9+\frac{1}{c}} \leq a+\frac{2020}{b+\frac{1}{c}}, \quad \text{and} \quad \frac{1}{a+\frac{2020}{9+\frac{1}{c}}} \geq \frac{1}{a+\frac{2020}{b+\frac{1}{c}}}$$

for any a and c. Since a and c are different, it follows that the greatest value occurs in one of the following two cases:

1) $a = 1$, $c = 2$; 2) $a = 2$, $c = 1$.

Let us check:

$$\frac{1}{1+\frac{2020}{9+\frac{1}{2}}} = \frac{19}{4059}, \quad \frac{1}{2+\frac{2020}{9+\frac{1}{1}}} = \frac{1}{204}, \quad \text{and} \quad \frac{19}{4059} < \frac{1}{204},$$

because $19 \cdot 204 = 3876 < 4059$.

5.49. If the numbers a and b are next to each other, then after them we must put the number $\frac{b}{a}$, after it, $\frac{1}{a}$, then, $\frac{1}{b}$, and, finally $\frac{a}{b}$. Since $a = \frac{a}{b} \cdot b$, each of the numbers will be equal to the product of its neighbours indeed. It remains to ensure that the six numbers obtained are all distinct. This is the case, for instance, if we take $a = 2$, $b = 3$.

5.50. *First solution.* After each Granddaughter's visit, the number of swedes is multiplied by $\frac{2}{3}$, after the dog's visit, the number of swedes is multiplied by $\frac{6}{7}$, while after the Mouse's visit, it is multiplied by $\frac{11}{12}$. Hence, the numbers of swedes in the patch cannot become divisible by 7 after the visit of anyone of these characters if it was not divisible by 7 before, and if it was divisible by 7 before, it can stop being divisible by 7 only after Zhuchka's visit. Since by the end of the week the number of swedes in the first patch was divisible by 7, the initial number of swedes was divisible by 7. At the beginning, the second patch contained as many swedes as the first one, while at the end only 4 remained. Hence at some moment of time the number of swedes there stopped being divisible by 7. Therefore, Zhuchka did visit the second patch.

Second solution. Since by the end of the week the number of swedes in the second patch was less than in the first one, the second patch was visited by some character. Let us find out who visited it last. If it were the mouse, then before its visit to the garden there would be $4 : \frac{11}{12} = 4\frac{4}{11}$ swedes. If it were the dog, then before the latter's visit there were $4 : \frac{6}{7} = 4\frac{2}{3}$ swedes. Since the number of swedes is always whole, the last visitor could only have been Granddaughter, and before her visit there were $4 : \frac{2}{3} = 6$ swedes in the garden. The number of swedes before Granddaughter's visit was smaller than 7, so someone must have visited the garden before her. Similarly one can see that this someone was the dog, and before its visit, there were $6 : \frac{6}{7} = 7$ swedes in the garden. Continuing this "reverse engineering", one can easily see that there were no more visits to the second patch, the initial number of swedes in each patch was 7, and that the first patch has not been visited at all.

5.51, 5.52, 5.53, 5.54, 5.55. See the answers section.

5.56. The search for an example is simplified if we notice that none of the denominators can be equal to 1 (in that case the denominators of the remaining fractions would coincide), nor to 5, nor to 7 (if the denominator of an irreducible fraction which is equal to the sum or difference of two fractions, is divisible by a prime number, then the denominator of at least one of these two fractions is divisible by the same prime number).

5.57. See the answers section.

5.58. The downriver part of the route took 1 hour, the upriver part, 3 hours. The 24 km route took 4 hours, hence the average speed was equal to $\frac{24}{4} = 6\text{km/h}$.

5.59. The sum of all ages of the players was $11 \cdot 22 = 242$, while after one player was sent off the field, they became equal to $10 \cdot 21 = 210$. Hence the age of the sent-off player is $242 - 210 = 32$.

5.60. The change of one grade by 24 points changes the average grade by 3 points. Hence the number of tests is $24 : 3 = 8$.

5.61. The arithmetic mean of the four numbers equals 10, hence the sum of these numbers equals 40. Similarly, the sum of three numbers (without the first) equals 33, the sum of three numbers (without the second) equals 36, while the sum of three numbers (without the third) equals 39. Therefore, the first number is 7, the second is 4, the third is 1. Thus, the arithmetic mean of the first three numbers is 4, and this is smaller by 6 than 10.

5.62. Let the sum of n numbers be equal to m. Then their arithmetic mean a is equal to $\frac{m}{n}$. If to this collection of numbers we add their arithmetic mean, then the sum of the new collection of numbers will be equal to

$$m + a = m + \frac{m}{n} = \frac{mn + m}{n} = \frac{m(n+1)}{n},$$

while their quantity will be $n + 1$. Hence their arithmetic mean will be

$$\frac{1}{n+1} \cdot \frac{m(n+1)}{n} = \frac{m}{n} = a.$$

Thus the arithmetic mean of the new collection is equal to the arithmetic mean of the initial collection.

5.63. The third shooter scored $(60 + 80) : 2 = 70$ points. Hence, each succeeding shooter also scored 70 points: if to the group of numbers we add the number which is the arithmetic mean of this group, then the arithmetic mean of the new group will become equal to the arithmetic mean of the initial group. (This is proved in the solution of Problem 5.62.)

5.64. Let us list what we know:

1) the sailor is 20;
2) the crew consists of six people ;
3) the helmsman is twice as old as the cabin boy;
4) the helmsman is 6 years older than the engineer;
5) the sum of the ages of the cabin boy and the engineer is twice that of the boatswain;

6) the boatswain is 4 years older than the sailor;

7) the mean age of the crew is 28.

Items 2 and 7 imply that the sum of ages of the whole crew equals $26 \cdot 6 = 168$. Items 1 and 6 imply that the boatswain is 24. Now item 5 implies that sum of ages of the cabin boy and the cockswain is 48. Conditions 3 and 4 imply that $2\mathrm{CB} = \mathrm{H} = \mathrm{E} + 6$. Adding the equations $\mathrm{CB} + \mathrm{E} = 48$ and $2\mathrm{CB} = \mathrm{E} + 6$, we obtain that the cabin boy is 18, hence the engineer is 30. Now item 3 implies that the helmsman is 36. Knowing the age of five members of the crew and the sum of ages of the whole crew, we find the age of the captain:

$$\mathrm{Ca} = 168 - (20 + 24 + 18 + 30 + 36) = 40.$$

5.65. Suppose that the arithmetic mean of the four given numbers is A. Then their sum is equal to $4A$. When the first number is deleted, the arithmetic mean increases by 1, hence the sum of the three remaining numbers is equal to $3A + 3$, i.e., the deleted number is smaller than A by 3. Similarly, the second number is smaller than A by 6, while the third number is smaller than A by 9. Hence, the fourth number is greater than A by $3 + 6 + 9 = 18$. If it is deleted, then the arithmetic mean will decrease by $18 : 3 = 6$.

5.66. Suppose that the numbers written on the board are $a < b < c < d < e < f < g$. Then Tania's number is

$$\frac{1}{7}(a + b + c + d + e + f + g),$$

while Daniel's number is d. Thus,

$$\frac{1}{7}(a + b + c + d + e + f + g) - d = \frac{3}{7},$$

i.e.,

$$a + b + c + e + f + g = 6d + 3.$$

The left-hand side of the obtained equation is the sum of six odd numbers, which is even, while the right-hand side is odd. This is impossible, hence either Daniel, or Tania, or both made a mistake.

5.67. One decorator does the job in $6 \cdot 5 = 30$ days, hence in 3 days the job will be done by $30 : 3 = 10$ decorators.

5.68. In $3 \cdot 5 = 15$ days one decorator can paint 60 windows, hence in one day they can paint $60 : 15 = 4$ windows. So in one day 5 decorators can paint $4 \cdot 5 = 20$ windows, while in 4 days they can paint $20 \cdot 4 = 80$ windows.

5.69. If we multiply the given fractions by 4, we will obtain the numbers 2, 3, and 6. Their sum equals 11, and $88 : 11 = 8$. The required parts equal $2 \cdot 8$, $3 \cdot 8$, $6 \cdot 8$.

5.70. We have $a : b = 3 : 2 = 15 : 10$ and $b : c = 5 : 3 = 10 : 6$. Hence $a : b : c = 15 : 10 : 6$. The sum of the numbers 15, 10, and 6 is 31, and $93 : 31 = 3$. Hence $a = 15 \cdot 3$, $b = 10 \cdot 3$, and $c = 6 \cdot 3$.

5.71. Before Tania arrived, Ivan had eaten four times as many pastries as Tania, i.e., twice as many as they both ate after Tania's arrival.

5.72. Half the peaches constitute one third of the volume of the can, i.e., half of the remaining two thirds of the liquid. Hence half of that half constitutes one fourth of the remaining liquid in the can.

5.73. One third of the peaches constitutes $\frac{1}{4}$ of the volume of the can, hence the remaining peaches constitute half the volume of the entire can. Therefore, the remaining peaches constitute $\frac{2}{3}$ of the remaining liquid.

5.74. The ratio $\frac{3}{2} : \frac{2}{3}$ equals $\frac{9}{4}$. The difference between the numbers 9 and 4 equals 5, and $25 : 5 = 5$. Hence the age of the father equals $9 \cdot 5$, while that of the son equals $4 \cdot 5$.

5.75. The vertical distance between Dolly's and Tania's flat is no more than 5 floors. The extra number of floors covered by Tania is equal to half the number of floors between their flats, hence it is at most two, and so the number of floors passed by Tania walking up is no greater than one. Therefore, Tania moved up by one floor.

5.76. Since 5 kg of coal fills the stove in the morning, and 7 kg, in the evening, if follows that during the stoker's wakefulness period, 7 kg of coal is burned (5 kg that were put in the morning, and an additional 2 kg that were in the stove before). In the evening, 7 kg is put in the stove, hence, while the stoker sleeps, 5 kg is burned (so that he can add exactly 5 kg in the morning). Thus, the moments of awakening and going to sleep divide the day into two parts, whose ratio is $7 : 5$, i.e., into 14 and 10 hours. Therefore the stoker is awake for 14 hours every day.

5.77. See the answers section.

5.78. The price of the first book constitutes 25 % of that of the second one, hence the second book is 4 times more expensive, i.e., by 300 %.

5.79. The same amount of potatoes can be bought for 80 % of the previous price. For the remaining 20 % of money, one can additionally buy one fourth of the previous purchase. So one can buy 25 % more potatoes.

5.80. The scholarship money of the A-student was multiplied by 2, that of the B-student by $\frac{3}{2}$. In order to get the B-student's scholarship from that of the A-student, one must multiply it by $\frac{3}{4}$, i.e., decrease it by 25 %.

5.81. *First solution.* $100 - 85 = 15$ % of all the students don't learn Greek, they only learn Latin. This means that $75 - 15 = 60$ % learn both languages.

Second solution. If we add 85 % and 75 %, we will get 100%, plus the percentage of those who were counted twice, i.e., $85 + 75 - 100 = 60$ % learn both languages.

5.82. The percentage of students who play basketball or tennis is $50 + 40 - 10 = 80$ %, hence 20 % of the students play neither one of these games.

5.83. After Dolly ate 30 % of the remainder of the pie, there remained $90 + 120 = 210$ g. Hence 210 g constitute 70 % of the remainder, and so before Dolly began eating, there was $210 : 0.7 = 300$ g. Before Peter began eating, there was $300 + 150 = 450$ g. Johnny ate 40 % of the pie, hence 450 constitute 60 % of it, and so the initial weight of the pie was equal to $450 : 0.6 = 750$ g.

5.84. Initially there was 1 % of dry substance in the berries. By the end of the storage, the weight of the same dry substance constituted 2 % of the weight of the berries. Hence, the weight of the berries decreased two-fold.

5.85. 1000 kg of grass contains $1000 \cdot \frac{100 - 60}{100} = 400$ kg of dry substance. The latter constitutes 80 % of the mass of hay. Hence the mass of hay is

$$\frac{100}{80} \cdot 400 = 500 \text{ kg.}$$

5.86. One Johnny's spoonful is 250 % (i.e., $\frac{250}{100} = \frac{5}{2}$) of Peter's spoonful. The number of spoonfuls eaten by Johnny is $\frac{3}{5}$ of the number of spoonfuls eaten by Peter. Thus Johnny ate $\frac{5}{2} \cdot \frac{3}{5} = \frac{3}{2}$ times more jam than Peter, so the jam was divided between then in a ratio $3 : 2$. Therefore, Johnny ate $\frac{3}{5}$ of jam.

5.87. *First solution.* By hypothesis $\frac{2}{100}A > \frac{3}{100}B$, so $A > \frac{3}{2}B$. We are to find out whether the inequality $A > \frac{7}{5}B$ is true. It is easy to verify that $\frac{3}{2} > \frac{7}{5}$. The number B is positive, hence $\frac{3}{2}B > \frac{7}{5}B$.

Second solution. The fact that 2 % of the positive number A is greater than 3 % of the positive number B implies that 4 % of the number A is greater than 6 % of the number B, and 1 % of the number A is greater than 1 % of the number B. Hence $4+1 = 5$ % of the number A is greater than $6+1 = 7$ % of the number B.

5.88. The second pirate took 51 % of the coins remaining after the first one, hence the third pirate got 49 % of this quantity. Therefore, 8 coins constitute 2 % of the coins remaining after the first pirate. Hence, together, the second and third pirates took $8 \cdot 50 = 400$ coins, which constitute $\frac{4}{7}$ of their total number. Thus, the sack contained $400 : \frac{4}{7} = 700$ coins.

5.89. 45 % constitute $\frac{45}{100} = \frac{9}{20}$ of the number of chocolates which remained before Johnny's lunch. Hence, this number is divisible by 20; denote it by $20n$. Note that $n < 6$, since $20 \cdot 6 = 120 > 111$. After the lunch, there remained $11n$ chocolates, while Matilda ate a third of this quantity. Hence n is divisible by 3. Therefore, $n = 3$, and Matilda ate 11 chocolates.

5.90. Suppose that the fat newspaper for pensioners costs $30k$ rubles (here $k < 1$), while the cost of the thin paper without discount is x rubles. By hypothesis, $kx = 15$, hence $30kx = 450$. Besides, we know that $30k$ and x are integers, hence x is a divisor of the number $450 = 2 \cdot 3^2 \cdot 5^2$, and $15 < x < 30$. Therefore, $x = 25$ or $x = 18$. The cost of the fat newspaper for pensioners is $30k = \frac{450}{x}$ rubles. In the first case, it costs 18 rubles, in the second case, it costs 25 rubles.

5.91. *First solution.* Suppose that last year there were 50 girls and 50 boys in school № 1, while in school № 2, there were 20 girls and 80 boys. Then the fraction of boys in the two schools together was

$$\frac{50+80}{200} = \frac{65}{100}, \quad \text{i.e., } 65\,\%.$$

Suppose that the same students go to school № 1 this year, while in school № 2 there is a greater number of pupils, say 40 girls and 160 boys.

Then the fraction of boys in the two schools together will increase to

$$\frac{50+160}{300} = \frac{70}{100}, \quad \text{i.e., to } 70\,\%.$$

Second solution. For this effect to take place, it suffices that the ratio of students in School № 2 to the number of students in both schools, would increase. Indeed, suppose that last year, as well as this year, in school № 1 there were n girls and n boys, while in school № 2 there were a girls and $4a$ boys, while last year there were b girls and $4b$ boys. Then last year the fraction of boys in the two schools was $\frac{n+4b}{2n+5b}$, and this year it is $\frac{n+4a}{2n+5a}$. If $a > b$, then

$$\frac{n+4a}{2n+5a} - \frac{n+4b}{2n+5b} = \frac{3n(a-b)}{(2n+5a)(2n+5b)} > 0.$$

5.92. If there were x A-students in the class and y new A-students were added, then

$$\frac{x+y}{32} - \frac{x}{25} = \frac{1}{10}.$$

Multiplying both sides of the equation by $32\cdot 25$, we obtain $25y-7x = 80$. Therefore, $25(y+1) = 7x+80+25 = 7(x+15)$. Hence, $y+1$ is divisible by 7. Since $y \le 7$, it follows that $y = 6$. Thus, $x = 10$, and the number of A-students became 16.

5.93. Use the fact that $\frac{27}{200} = 0.135$.

5.94. Use the fact that $\frac{3}{4} = 0.75$ and $\frac{37}{50} = 0.74$.

5.95. Converting the fractions to fractions with same denominator, check that

$$\frac{4}{10} < \frac{3}{7} \quad \text{and} \quad \frac{3}{7} < \frac{6}{13}.$$

5.96. Placing a decimal point between 2 and 3 we obtain the number 2.3. Clearly, $2 < 2.3 < 3$.

5.97. 8 sandwiches, 2 cups of coffee, and 20 doughnuts cost $2\cdot 1.69 = 3.38$ dollars, while 9 sandwiches, 3 cups of coffee, and 21 doughnuts cost $3 \cdot 1.26 = 3.78$ dollars. The difference between these prices is equal to the cost of one sandwich, one cup of coffee, and one doughnut.

5.98. Both $0.2 \cdot 10 = 2$ and $0.2 \cdot 15 = 3$ are prime numbers.

5.99. See the answers section.

5.100. Suppose the denominator of the fraction is $2^n \cdot 5^m$, where m and n are natural numbers. Denote the largest of these numbers by k and multiply the numerator and denominator of the given fraction by $2^{k-n} \cdot$

5^{k-m}. As a result, we obtain a fraction with denominator 10^k. Note that a fraction with denominator of the form 2^n or 5^m, where m and n are natural numbers, can also be represented in the form of a decimal fraction in a similar way.

5.101. Suppose that an irreducible fraction $\frac{p}{q}$ can be represented as a decimal fraction. Then

$$\frac{p}{q} = \frac{m}{10^n},$$

where m and n are natural numbers. So, $p \cdot 10^n = qm$. The numbers p and q have no common divisors, hence q is a divisor of the number 10^n. Therefore, q cannot be divisible by a prime number different from 2 and 5.

5.102. The salary will be multiplied by $1.2 \cdot 0.8 = 0.96$, i.e., it will decrease by 4 %.

5.103. In the second store, the new price is obtained from the old one by multiplying by $0.8 \cdot 0.75 = 0.6$. In the first store, the new price is also obtained from the old one by multiplication by 0.6.

5.104. The area was multiplied by $1.5 \cdot 0.9 = 1.35$, i.e., it increased by 35 %.

5.105. Johnny's weight was multiplied by $1.2 \cdot 0.9 \cdot 1.2 \cdot 0.75 = 0.972$. Thus his weight decreased by 2.8 %.

5.106. When half the clerks have been fired, the expenditures will be cut by half. After that the salaries will be increased 1.5 times. As a result, their total value will constitute $1.5 \cdot 0.5 = 0.75$ of their previous value, i.e., the total value will decrease by 25 %.

5.107. After two years the output was $0.49 = 0.7^2$ of the initial one. So each year it was 0.7 of the previous one, i.e., it decreased by 30 %.

5.108. Let us compare everything with Basil's initial amount of water. Initially Peter had 1.1 times that amount. After each of them drank from their bottle, Basil still had 0.98, while Peter had $0.89 \cdot 1.1 = 0.979$.

5.109. In the first glass, the amount of water was multiplied by

$$1.27 \cdot 1.26 \cdot \ldots \cdot 1.01,$$

while in the second glass, it was multiplied by

$$1.01 \cdot 1.02 \cdot \ldots \cdot 1.27.$$

These products are equal.

5.110. It follows from the conversion rule that 1 pound equals $0.5 - 0.05 = 0.45$ kg. Converting 0.45 kg to pounds by his method, Joe will get $0.9 + 0.09 = 0.99$ pounds, which is 1 % less than 1 pound.

5.111. After the first washing, the volume of the piece of soap is 0.8 of the initial one, after the second it is 0.64, after the third, 0.512, after the fourth, 0.4096.

5.112. Suppose n chess players participate in the tournament and k of them are grandmasters. According to the hypothesis, $0.9n < 2k < n$, hence $0.1n > n - 2k > 0$. The number $n - 2k$ is whole, hence it is no less than 1. Therefore, $0.1n > 1$, i.e., $n > 10$. The number $n = 11$ will do: there can be 6 chess masters and 5 grandmasters.

5.113. The increase of a number by 20 % is equivalent to its multiplication by 1.2, while its decrease by 20 % is equivalent to its multiplication by 0.8 (for 30 % this is equivalent to its multiplication by 1.3 and 0.7, respectively). Hence the result does not depend on the alternation of good or bad weather, it depends only on the number of good and bad days. After one good day and one bad day, both bullions decrease: $1.2 \cdot 0.8 = 0.96 < 1$ and $1.3 \cdot 0.7 = 0.91 < 1$. Hence after three good and three bad days, both bullions decrease. Therefore, there were at least four good days. If the number of good days was 4 and the number of bad days was 3, the gold bullion decreases ($1.3^4 \cdot 0.7^3 = 0.9796423 < 1$), while the silver one increases ($1.2^4 \cdot 0.8^3 = 1.0616832 > 1$). After two good days and one bad one, both bullions increase ($1.2^2 \cdot 0.8 = 1.152 > 1$ and $1.3^2 \cdot 0.7 = 1.183 > 1$). Hence, they also increase after four good and two bad days and all the more, after five good and two bad days. Thus, if there are five good days or more, then both bullions increase. Hence, it is only after four good days that one bullion increases, while the other decreases.

5.114. Increase by 10 % means multiplication by 1.1, decrease by 10 % means multiplication by 0.9. Since $8019 = 9 \cdot 9 \cdot 9 \cdot 11$, after three misses and one hit, Johnny will have $10000 \cdot (0.9)^3 \cdot 1.1 = 8019$ cents.

5.115. Suppose Boris gathered x mushrooms. Then Alex gathered $0.8x$, while Basil gathered $1.2x$ mushrooms. Hence Basil gathered $1.2 : 0.8 = 1.5$ times more mushrooms than Alex.

5.116. *First solution.* Suppose there were x coins in the chest. The first pirate took $0.3x$, after which $0.7x$ coins remained. The second took $0.7 \cdot 0.4x = 0.28x$, after which $0.7x - 0.28x = 0.42x$ remained. The third took $0.42 \cdot 0.5x = 0.21x$ coins, and the same number of coins remained.

We know that the number of remaining coins is 63, so $0.21x = 63$, and $x = 300$.

Second solution. After the first and second pirates, there remained

$$63 \cdot \frac{100}{100-50} = 126 \text{ coins.}$$

Hence, after the first pirate, there remained

$$126 \cdot \frac{100}{100-40} = 210 \text{ coins,}$$

while initially there were

$$210 \cdot \frac{100}{100-30} = 300 \text{ coins.}$$

5.117, 5.118. See the answers section.
5.119. Cut each of the two other apples in half and use the identity from Problem 5.118.
5.120. See the answers section.
5.121. Cut four apples in half, two apples, in four parts, and use the identity from Problem 5.120.
5.122. It follows from the identity

$$1 = \frac{6}{6} = \frac{1+2+3}{6}.$$

5.123. One can write the following chain of identities:

$$1 = \frac{1}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}.$$

5.124. The equation $\frac{1}{3} = \frac{1}{p} + \frac{1}{q}$ can be written in the form $p = 3 + \frac{9}{q-3}$. For $q = 4$, we obtain $p = 12$.
5.125. See the answers section.
5.126. Use the following identities:

$$\frac{5}{8} = \frac{1+4}{8} = \frac{1}{8} + \frac{1}{2}, \quad \frac{7}{10} = \frac{2+5}{10} = \frac{1}{5} + \frac{1}{2},$$
$$\frac{5}{6} = \frac{3+2}{6} = \frac{1}{2} + \frac{1}{3}, \quad \frac{3}{5} = \frac{6}{10} = \frac{5}{10} + \frac{1}{10} = \frac{1}{2} + \frac{1}{10},$$
$$\frac{4}{5} = \frac{8}{10} = \frac{5}{10} + \frac{2}{10} + \frac{1}{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}, \quad \frac{2}{5} = \frac{1}{2} \cdot \frac{4}{5}.$$

5.127. Suppose that $m > n$ and $\frac{1}{m} + \frac{1}{n} = \frac{1}{25}$. Then

$$m = \frac{25n}{n-25} = 25 + \frac{625}{n-25}.$$

If $n - 25 = 1$, then $m = 650$.
If $n - 25 = 5$, then $m = 150$.
If $n - 25 = 25$, then $m = n$.

5.128. Suppose that $m > n$ and $\frac{1}{m} + \frac{1}{n} = \frac{1}{6}$. Then

$$m = \frac{6n}{n-6} = 6 + \frac{36}{n-6}.$$

If $n - 6 = 1$, then $m = 42$.
If $n - 6 = 2$, then $m = 24$.
If $n - 6 = 3$, then $m = 18$.
If $n - 6 = 4$, then $m = 15$.
If $n - 6 = 6$, then $m = n$.

5.129. As in the solutions of Problems 5.127 and 5.128, we see that the number of representations is equal to the number of divisors of the number $12^2 = 144$ which are less than 12. To the divisors 1, 2, 3, 4, 6, 8, and 9 correspond the representations

$$\frac{1}{13} + \frac{1}{156}, \ \frac{1}{14} + \frac{1}{84}, \ \frac{1}{15} + \frac{1}{60}, \ \frac{1}{16} + \frac{1}{48}, \ \frac{1}{18} + \frac{1}{36}, \ \frac{1}{20} + \frac{1}{30}, \ \frac{1}{21} + \frac{1}{28}.$$

5.130. See the answers section.

5.131. The sum of these fractions is equal to

$$\frac{bcd + acd + abd + abc}{abcd}.$$

The numerator of this fraction is the sum of four odd numbers. Hence it is an even number, while the denominator is odd. Such a fraction cannot be equal to 1.

5.132. Suppose $\frac{m}{n}$ is a proper fraction, $m > 1$. Write n in the form $n = qm - r$, where $0 \leq r < m$. Then

$$\frac{m}{n} = \frac{qm}{qn} = \frac{n + r}{qn} = \frac{1}{q} + \frac{r}{qn}.$$

For $r = 0$, we obtain the required representation. Now if $r \neq 0$, then we repeat the same calculation for the proper fraction $\frac{r}{qn}$, and so on.

In the obtained representation of the fraction $\frac{m}{n}$, there are no identical summands, since

$$\frac{r}{qn} < \frac{m}{qn} < \frac{n}{qn} = \frac{1}{q}.$$

Moreover, since $r < m$, the numerator of the fraction $\frac{r}{qn}$ is less than that of the original fraction, so after a finite number of such steps we will obtain the desired representation.

Chapter 6. Integers and Rational Numbers

6.1. Clearly, all these numbers equal ± 1. Their sum can equal zero only if there is an equal number of plus ones and minus ones, namely 11. But then their product would equal -1.

6.2. The identity $\dfrac{n+9}{n+6} = 1 + \dfrac{3}{n+6}$ shows that the given number is whole if and only if $n+6 = \pm 1$ or ± 3.

6.3. Each of the products is either 1 or -1, and the number of these products, which is 25, is odd. The sum of an odd number of 1's and -1's cannot be zero (cf. the solution of Problem 6.1).

6.4. Since each "row product" (the product of numbers in a given row) is negative and the total number of rows is odd, the product of all row products, which is equal to the product of all numbers in the table, is negative. But the latter number is equal to the product of all "column products" as well. Hence, at least one of the column products must be negative.

6.5. The product of all the summands is $-(abcdefghk)^2$, so it is negative. Hence, the number of summands that are equal to -1 is odd; in particular, it is nonzero. The sum of six 1's or -1's is even. Therefore, the value of the expression under consideration is at most 4. This value is attained if $a = c = d = e = g = h = k = 1$ and $b = f = -1$.

6.6. See the answers section.

6.7. Represent the product $2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2018 \cdot 2020$ as

$$(2021 - 2019) \cdot (2021 - 2017) \cdot (2021 - 2015) \cdot \ldots \cdot (2021 - 3) \cdot (2021 - 1)$$

and expand all the brackets. All the resulting terms save one will contain the factor 2021. The only term that does not contain this factor is

$$1 \cdot 3 \cdot 5 \cdot \ldots \cdot 2017 \cdot 2019$$

(the sign is plus since it is the product of 1010 negative factors). After subtracting $1 \cdot 3 \cdot \ldots \cdot 2019$, only the terms divisible by 2021 remain.

6.8. The numbers 53 and -96, 83 and -66, 109 and -40 have the same remainders under division by 149, because the differences of these pairs of numbers all equal 149. Hence the numbers $53 \cdot 83 \cdot 109$ and $-40 \cdot 66 \cdot 96$ have the same remainders under division by 149. Therefore their difference, which equals

$$53 \cdot 83 \cdot 109 + 40 \cdot 66 \cdot 96,$$

is divisible by 149.

6.9. The value of each coin has remainder 1 under the division by 7. The price of the purchase is the difference between what Cyrano gave and what he received as change. He received one more coin than he gave, hence the difference between the remainders under the division by 7 is equal to -1 or 6, so the remainder of the difference under the division by 7 equals 6. Therefore, the price of the purchase has the remainder 6 under the division by 7, hence the price of the purchase cannot be less than 6 farthings. If Cyrano gave a 1-farthing coin and a 50-farthing coin and received change in three 15-farthing coins, then the price of the purchase was exactly 6 farthings.

6.10. If $ab < 0$, then one of the numbers a and b is positive, while the other is negative. Assume that $a > 0$ and $b < 0$. Then $b = -c$, where $c > 0$, and $|a+b| = |a-c|$, $|a|+|b| = a+c$. The sum of positive numbers is greater than modulus of their difference, hence $|a + b| < |a| + |b|$. If $a < 0$ and $b > 0$, the proof is similar.

6.11. a) If $ab < 0$, then the numbers a and b have opposite signs and so cannot be equal.

b) If $ab > 0$, then the numbers a and b are either both positive, or both negative. In the first case, the equality $|a| = |b|$ implies that $a = b$, in the second case it implies that $-a = -b$.

6.12. Suppose that the fractions are $\frac{x}{8}$ and $\frac{y}{13}$. Then the difference between the greater and the smaller one is equal to $\frac{|13x - 8y|}{104}$. The numerator of the obtained fraction is a natural number, so the fraction is at least $\frac{1}{104}$. This value is attained if, say, $x = 3$ and $y = 5$.

6.13. Note that $|36^1 - 5^2| = 11$; let us prove that $|36^k - 5^l| \geq 11$ for any k and l. Observe that the number $|36^k - 5^l|$ is odd and is not divisible by 3 nor by 5, hence it cannot equal 0, 2, 3, 4, 5, 6, 8, 9, or 10. It cannot be equal to 7 either since its remainder under the division by 5 is 1 or 4. Let us show that $|36^k - 5^l|$ cannot equal 1. Indeed, if $36^k - 5^l = 1$, then the number $5^l = 36^k - 1$ must be divisible by $36 - 1 = 35$ (the result of division equals $36^{k-1} + 36^{k-2} + \ldots + 36 + 1$), which is impossible. If $36^k - 5^l = -1$, then the last digit of the number $5^l = 36^k + 1$ must be 7, which is also impossible.

6.14, 6.15, 6.16. See the answers section.

6.17. Since $\frac{1}{14} = \frac{1}{10} \cdot \frac{5}{7}$, the representation of the fraction $\frac{1}{14}$ is obtained from the representation of the fraction $\frac{5}{7}$ by multiplication by 0.1, i.e., adding a zero before the periodic part.

6.18. See the answers section.

6.19. Inserting a zero immediately after the decimal point corresponds to the multiplication by $\frac{1}{10}$, inserting two zeros, to multiplication by $\frac{1}{100}$.

6.20. *First solution.* This result can be obtained by long division, since

$$10 = 9 + 1, \quad 100 = 99 + 1, \quad 1000 = 999 + 1, \quad \text{and so on.}$$

Second solution. First write the identity

$$\frac{1}{9} = \frac{10}{90} = \frac{9+1}{90} = \frac{1}{10} + \frac{1}{10} \cdot \frac{1}{9},$$

then, in the right-hand side, replace $\frac{1}{9}$ by $\frac{1}{10} + \frac{1}{10} \cdot \frac{1}{9}$ and so on. For $\frac{1}{99}$ and $\frac{1}{999}$, we can continue similarly by using the identities

$$\frac{1}{99} = \frac{99+1}{9900} = \frac{1}{100} + \frac{1}{100} \cdot \frac{1}{99} \quad \text{and} \quad \frac{1}{999} = \frac{999+1}{999\,000} = \frac{1}{1000} + \frac{1}{1000} \cdot \frac{1}{999}.$$

6.21. According to Problem 6.20, one has $0.(01) = \frac{1}{99}$ and $0.(001) = \frac{1}{999}$. Hence,

$$0.(13) = 13 \cdot 0.(01) = \frac{13}{99} \quad \text{and} \quad 0.(238) = 238 \cdot 0.(001) = \frac{238}{999}.$$

6.22. Let us represent the infinite periodic fractions as proper fractions as in Problem 6.21.

a) $0.(3) = \frac{3}{9} = \frac{1}{3}$ and $0.(4) = \frac{4}{9}$, hence

$$0.(3) \cdot 0.(4) = \frac{1}{3} \cdot \frac{4}{9} = \frac{4}{27}.$$

To obtain an infinite periodic fraction, we need a fraction whose denominator consists of repeating nines. Since $999 : 27 = 37$, one has $\frac{4 \cdot 37}{999} = \frac{148}{999} = 0.(148)$.

b) Clearly, $0.(12) = \frac{12}{99}$ and $0.(122) = \frac{122}{999}$. Hence,

$$0.(12) + 0.(122) = \frac{12}{99} + \frac{122}{999}.$$

Both the denominators in the right-hand side are divisors of 999 999. Converting both fractions to this common denominator, one sees that the required sum equals

$$\frac{121\,212 + 122\,122}{999\,999} = \frac{243\,334}{999\,999} = 0.(243\,334).$$

6.23. As in Problem 6.21, we obtain

$$0.(23) = \frac{23}{99} \quad \text{and} \quad 0.(2323) = \frac{2323}{9999} = \frac{23 \cdot 101}{99 \cdot 101} = \frac{23}{99}.$$

6.24. As in Problem 6.21, we obtain

$$0.(9) = \frac{9}{9}, \quad 0.(99) = \frac{99}{99} \quad \text{and so on.}$$

6.25. The number 1 can be written as 0.(99), so

$$1 - 0.(85) = 0.(99) - 0.(85) = 0.(14) = \frac{14}{99}.$$

6.26, 6.27, 6.28, 6.29, 6.30. See the answers section.

6.31. The six numbers can be divided into three pairs of neighbouring numbers. If the sum of the numbers in each pair of neighbouring numbers is positive, then the sum of all six numbers must be also positive.

6.32. The nine numbers can be divided into three triplets of neighbouring numbers. If the sum of the numbers in each triplet is positive, then the sum of all nine numbers is also positive.

6.33. a) Eighteen numbers can be divided into six triplets of consecutive numbers. The sum of the numbers in each triplet is positive, hence the sum of all the numbers is positive.

b) Consider, for instance, the numbers $-7, -7, 15, \ldots, -7, -7, 15, -7$ (6 identical triplets $(-7, -7, 15)$ and then 7). The sum of numbers in each triplet of consecutive numbers equals 1, the sum of all the numbers equals -1.

c) Consider, for instance, the numbers -7, -7, 16, …, -7, -7, 16, -7, -7 (6 identical triplets $(-7, -7, 16)$ and then -7 twice). The sum of the consecutive numbers in each triplet equals 2, the sum of all the numbers equals -2.

6.34. If this were possible, then the sum of all the numbers in the table, counted via the rows, would be positive, while the same sum counted via the columns would be negative.

6.35. a) One can place -5 in each cell.

b) If the sum of numbers in each row is -20, then the sum of all the numbers in the table is -60. On the other hand, if the sum of numbers in each row is -16, then the sum of all the numbers in the table is -64.

6.36, 6.37. See the answers section.

Chapter 7. Equations

7.1. Denote the required number by x and write the equation $4x+15=15x+4$. Solving the equation, we obtain $11=11x$, i.e., $x=1$.

7.2. *First solution.* Suppose that initially there were x Republicans and x Democrats, and then this changed to $x-3$ Republicans and $x+3$ Democrats, which is twice as many as $x-3$. So, $x+3=2(x-3)$, hence $x=9$.

Second solution. The number of Democrats became larger than the number of Republicans by 6, and this difference is equal to the number of Republicans. Therefore, the number of Republicans became equal to 6 and the number of Democrats became equal to 12.

7.3. Suppose the first number is 100 and the second is x. After the increase, these numbers will become 101 and $1,04x$. From the equation $101+1.04x=1.03(100+x)$, we find $x=200$.

7.4. Suppose Johnny worked for x days. Then $48x=12(30-x)$, i.e., $4x=30-x$. Hence $5x=30$ and $x=6$.

7.5. *First solution.* Suppose that there are x, y, and z kg of apples in the first, second, and third box, respectively. We know that $x=y+z-6$ and $y=x+z-10$. Adding these equations, we get $0=2z-16$, i.e., $z=8$.

Second solution. The statement implies that the mass of apples in the first and the second boxes is $6+10=16$ kg less than the mass of apples in the first box, the second box, and two identical copies of the third box. Hence there are 16 kg of apples in two copies of the third box, i.e., the third box contains 8 kg of apples.

7.6. *First solution.* Suppose mother assigned x branches and y decorations to each child. It follows from what we know that $x=y-1$ and $x=\frac{y}{2}+1$. Therefore, $y-1=\frac{y}{2}+1$, i.e., $y=4$ and $x=y-1=3$.

Second solution. Let us hang, like John did, one toy on each branch. Then one of John's toys will be left. Now rearrange the toys on John's branches as follows. Take this remaining toy and a toy from one of John's branches and hang these two as second toys on two of John's branches on which one decoration hangs already. Then on two branches two toys will be hanging, and one branch will be empty, which is precisely the situation with Mary's toys and branches. Since both children were assigned an equal number of branches and given an equal number of toys, one concludes that 3 branches were assigned to each of them and each of them was given 4 toys.

7.7. Suppose Johnny had J remaining pieces, and Freddy, F. We know that $J = 2F$ and (since there are 64 squares on the board) $5F = 64 - F - J$, so that $64 = 5F + F + J = 5F + F + 2F = 8F$. Thus Johnny had $J = 64 : 8 = 8$ pieces, while Freddy had all his 16.

7.8. Suppose George hung up x toys, then Mary hung up $2x$ toys, and together they hung up $3x$ toys. Ilona hung up $2 \cdot 3x = 6x$ toys. She hung up 15 toys more than George, hence $6x - x = 15$, and so $x = 3$. Thus there 27 toys hanging on the Christmas tree.

7.9. Denote the ratio of lengths of the arms of the scales by k. If x is the actual weight of the sack, then $xk = 50$ and $x = 32k$, which implies that $k = \frac{50}{x} = \frac{x}{32}$. Hence $x^2 = 32 \cdot 50$ and $x = 40$, i.e., the true weight of the sack is 40 kg.

7.10. Suppose that one kilogram of beef with bones contains x kg of bones; then it contains $(1-x)$ kg of pure beef. Thus, $15x+90(1-x) = 78$, and therefore $x = 0.16$.

7.11. If the number at one of the endpoints of the first side is x, then the number at the other endpoint of this side is $1-x$, the number at the next vertex is $2 - (1 - x) = 1 + x$, the number at the fourth vertex is $3-(1+x) = 2-x$, and the number at the fifth vertex is $4-(2-x)2+x$. Since the sum of numbers at the first and at the fifth vertex must be 5 one has $x + (2 + x) = 5$. From this equation we find $x = \frac{3}{2}$.

7.12. Suppose that the King is t years older than the Queen, and let x be the age of the Queen at the moment when the King was as old as she is today. Then the Queen is $x + t$ years old now, and the King is $x + 2t$ years old. On the other hand, now the King is twice older than the queen, so he is $2x$ years old. Hence $x + 2t = 2x$, i.e., $x = 2t$, so the King's age equals $4t$. When the Queen will be $4t$ years old, the King will be $5t$. Their ages will add up to $9t$, which must be equal to 63. Hence $t = 7$.

7.13. Suppose the speed of the truck is x km/h. Then in 10 hours it will cover $10x$ km. If the speed is increased by 10km/h, the truck will cover $8(x + 10)$ km in 8 hours. Hence $10x = 8(x + 10)$, so $x = 40$.

7.14. Denote the actual speed of the "A-car" by x km/h and that of the "B-car" by y km/h. Then in 8 hours together they have covered $8(x+y)$ km, while after the increase of speed they would cover $7(1.14x+1.15y)$ km. The equation $8(x + y) = 7(1.14x + 1.15y)$ implies $0.02x = 0.05y$, i.e., $x = 2.5y$.

7.15. Suppose that the required distance is x km. Then the speed of one car is $\frac{x}{10}$ km/h and that of the other is $\frac{x}{12}$ km/h. The sum of distances covered by the cars in 7 hours is

$$7\left(\frac{x}{10}+\frac{x}{12}\right)=\frac{77x}{60} \text{ km.}$$

That distance is greater than the distance between A and B. This means that the cars met and moved away from each other by

$$\frac{77x}{60}-x=\frac{17x}{60} \text{ km.}$$

Thus $\frac{17x}{60}=136$, so $x=480$.

7.16. Suppose that the whole distance is S km and the initial speed of the motorboat is v km/h. Then

$$\frac{S}{2v}-\frac{S}{2\cdot 1.25v}=\frac{1}{2}$$

Hence $\frac{S}{v}=5$, i.e., the trip at the initial speed would have taken 5 hours, so the actual trip took $5-0.5=4.5$ hours.

7.17. In 20 minutes (one third of an hour), the minute hand of the clock turns by one third of a full rotation, i.e., by $120°$, while the hour hand only turns by $10°$, since it moves 12 times slower. During this 20 minute interval, the minute hand either overtakes the hour hand, or not. Denote the initial angle between the hands by α. If the minute hand has overtaken the hour hand, then $2\alpha+10°=120°$, and so $\alpha=55°$. If it has not overtaken the hour hand, then $2\alpha-10°+120°=360°$, so that $\alpha=125°$.

7.18. Suppose the time difference between the city A and the city B is x hours and the flight lasted y hours. Then $y-x=2$ and $y+x=6$. Adding these equalities, we see that $y=4$.

7.19. Suppose that today I covered x km, and yesterday, y km. Then the day before yesterday I covered $(y+3)$ km. Hence, $y+40=(y+3)+x$, and so $x=37$.

7.20. Suppose that Pooh's speed is x m/min, Piglet's speed is y m/min, and the elapsed time between the moment when they started and the moment they crossed each other is t min. Then the distance from the point where they crossed each other to Piglet's house is yt m, and the distance to Pooh's house is xt m. Hence $\frac{yt}{x}=4$ and $\frac{xt}{y}=1$. Multiplying these equations, we obtain $t^2=4$ and $t=2$.

7.21. Suppose that the distance between A and B is x km, the speed of the truck that started at A is v km/h, while the speed of the truck that started at B is w km/h. At the moment of their first meeting, the trucks had covered 60 km and $(x-60)$ km, hence $\frac{60}{v} = \frac{x-60}{w}$. At the moment of their second meeting, they had covered $(x+80)$ km and $x + (x - 80) = (2x - 80)$ km, hence

$$\frac{x+80}{v} = \frac{2x-80}{y}.$$

Therefore, $x = \dfrac{60(v+w)}{v}$ and $x = \dfrac{80(v+w)}{2v-w}$. Equalizing these two expressions for x, we get $3w = 2v$. Hence

$$x = 60\left(\frac{w}{v}+1\right) = 60 \cdot \frac{5}{3} = 100.$$

7.22. In 24 hours the number of lotuses doubles. Hence 24 hours before the lake is filled with lotuses, it will be exactly half covered. Thus the lake will be half filled with lotuses on the 29th day.

7.23. Kittycat replaces 6 mice. The dog Zuchka replaces $5 \cdot 6 = 30$ mice. Granddaughter replaces $4 \cdot 30 = 120$ mice. Grandma replaces $3 \cdot 120 = 360$ mice. Grandpa replaces $2 \cdot 360 = 720$ mice. Thus a total of $720 + 360 + 120 + 30 + 6 + 1 = 1237$ mice are required.

7.24. Victor took 4 plums, which means that he saw $4 \cdot 3 = 12$ plums. Hence, Boris took 6 plums, having seen 18. Anna took 9 plums, and mother left 27.

7.25. Let's start with the third son. Half of the new remainder consists of half an apple, so the third son got one apple. After the second son, there remained half of the remainder minus half an apple, and this constitutes one apple. Hence the remainder is three apples. The remainder is half of all the apples minus half an apple. Hence there were 7 apples in all.

7.26. Suppose half of x lemons were taken together with half a lemon, and a lemons remained. Then $x - \frac{x}{2} - \frac{1}{2} = a$. Hence 31 lemons were obtained from $2 \cdot 31 + 1 = 63$ lemons, 63 were obtained from $2 \cdot 63 + 1 = 127$ lemons, while 127 lemons were obtained from $2 \cdot 127 + 1 = 255$ lemons.

7.27. Suppose one third of x plums plus 8 plums were taken, and after that a plums remained. Then $x - \frac{x}{3} - 8 = a$, i.e., $x = \frac{3a}{2} + 12$. After the third sister nothing remained ($a = 0$), and so she got 12 plums. The second sister took her share out of $\frac{3 \cdot 12}{2} + 12 = 30$ plums, so she took

$\dfrac{30}{3}+8=18$ plums. The third sister took her share out of $\dfrac{3\cdot 30}{2}+12=$ 57 plums, obtaining $\dfrac{57}{3}+8=28$ plums.

7.28. Suppose the box initially contained x matches, and after their number was doubled and 8 matches were removed, a matches remained. Then $2x-8=a$, i.e. $x=\dfrac{8+a}{2}$. Hence before the third operation the box contained $\dfrac{8}{2}=4$ matches, before the second operation $\dfrac{8+4}{2}=$ 6 matches, and before the first, $\dfrac{8+6}{2}=7$ matches.

7.29. Initially the man had $((24:2+24):2+24):2=21$ dollars.

7.30. In the last game, the first pirate lost half of his coins, and 15 coins remained. He gave the same amount to the second pirate, hence before that the first pirate had $15\cdot 2=30$ coins, while the second had $33-15=18$ coins. In the second game, the second pirate lost exactly as much as he had after the second game, i.e., 18 coins. Hence the second pirate had $18\cdot 2=36$ coins, while the first had $30-18=12$ coins. Thus, in the first game, the first pirate gave away 12 coins, so before the game began he had $12\cdot 2=24$ coins (the second pirate, incidentally, also had $36-12=24$ coins).

7.31. Let us simplify the equation from outside the brackets rather than inside them:

$$\frac{8}{1992}=1+8:(1-8:(1+4:(1-4:(1-8:x)))),$$
$$-\frac{249}{31}=1-8:(1+4:(1-4:(1-8:x))),$$
$$\frac{31}{35}=1+4:(1-4:(1-8:x)),\quad -35=1-4:(1-8:x),$$
$$\frac{4}{36}=1-8:x,\quad x=9.$$

7.32. The difference between a number and the sum of its digits is divisible by 9, hence all the numbers obtained by the successive subtractions are divisible by 9. If a number has more than one digit, then the difference between the number and its sum of digits is strictly positive; thus, since the eleventh number is zero, the tenth number is 9.

Moreover, if a number has three or more digits, then the difference between the number and its sum of digits is at least 99, and if a number has two digits, then the difference between it and its sum of digits is 9 times the digit in the second position (the tens place). Taking into account that all the numbers obtained by subtractions are divisible by 9, one sees that, since the tenth number was 9, the ninth number was 18, the eighth number was 27, ..., the second number was 81. Hence, in the second number the digit in the tens place must be $81/9 = 9$; since this number is also divisible by 9, it can be either 90 or 99.

If the second number is 90, we arrive at a contradiction since in this case the first (original) number must be two-digit. Hence, the second number is 99 and the original number is three-digit. If a, b, and c are its digits at the hundreds, the tens, and the units places respectively, then

$$99 = (100a + 10b + c) - (a + b + c) = 99a + 9b,$$

so $a = 1$, $b = 0$, and c may be arbitrary.

7.33. If, at some step, we arrived at the configuration of nine zeroes, then at the previous step zeroes and ones alternated, which is impossible since one cannot place in a circle an odd number of alternating zeroes at ones. If we arrived at the configuration of nine ones, then its first occurrence was preceded by the configuration of nine zeroes, which, as we have shown, is impossible.

7.34. If x and y are the digits at the tens and units places respectively, then $10x + y = 2(x + y)$, i.e., $8x = y$. Since y is a digit, the only answer is 18.

7.35. If a and b ar he digits at the tens and units places respectively, then $10a + b = 2ab$. This equation can hold only if b is even and $b \neq 0$. Putting $b = 2c$, we obtain $5a + c = 2ac$ or $5a = (2a - 1)c$. Since $b < 10$, we have $c < 5$. Hence $2a - 1$ is divisible by 5. Thus $a = 3$ or $a = 8$. But $5 \cdot 8$ is not divisible by $2 \cdot 8 - 1$. Therefore, $a = 3$, $c = 3$, and $b = 6$.

7.36. The number of fish caught by the first fisherman is divisible by 9, and that of the second fisherman, by 17. Subtracting successively 17 from 70, we obtain 53, 36, 19, and 2. Among these numbers only 36 is divisible by 9.

7.37. The catch of the first fisherman is divisible by 9, and that of the second one, by 11.

First solution. Subtracting successively 11 from 80, we obtain 69, 58, 47, 36, 25, 14, 3. Among these numbers only 36 is divisible by 9. Hence the first caught 36 fish, the second, 44.

Second solution. If the first (resp. the second) fisherman caught a (resp. b) fish, then $80 = 9a + 11b$, where a and b are natural numbers. Hence $2(b - 4) = 2b - 8 = 9(8 - b - a)$ is divisible by 9. Since $b < 8$, we have $b = 4$. Hence the first caught 36 fish, the second, 44.

7.38. Suppose n two-kilogram weights and m five-kilogram weights were placed on the scales. Then $2n = 5m$, hence $n = 5k$ and $m = 2k$ for some integer k. Hence $5k + 2k = 14$, so $k = 2$.

7.39. The total volume of the 17-litre cans, in litres, ends in the digit 3, hence the number of such cans ends in 9. This number cannot be more than 9, because $17 \cdot 19 > 223$. Thus there are nine 17-litre cans, they contain $17 \cdot 9 = 153$ litres, while the other $223 - 153 = 70$ litres are contained in the 10-litre cans. So, there are $9 + 7 = 16$ cans in all.

7.40. Suppose there are x pears. Then there are $9x$ peaches and $20 - x - 9x = 10(2 - x)$ apples. Since we know that there is at least one pear and at least one apple, $x > 0$ and $2 - x > 0$, hence $x = 1$.

7.41. The condition $A \cdot C = C$ implies that $A = 1$. Since $B \cdot B = 10 + C$, where C is a digit, one has $B = 4$ and $C = 6$.

7.42. Suppose x candies in all were eaten, and one of the children ate y candies. Then all the others ate $x - y$ candies, so $(x - y) - 7 = y$, i.e., $2y = x - 7$. Therefore, each child ate one and the same number of candies, namely $\frac{x-7}{2}$. If the number of children was n, then $x = ny$, so the equation $2y = x - 7$ implies $2y = ny - 7$, i.e., $(n - 2)y = 7$. By condition, $y > 1$, hence $y = 7$ and $x = 2y + 7 = 21$.

7.43. Suppose n is the number of cups that were consumed (which coincides with the number of people in the family), while x is the amount of consumed milk (measured in cups). Then the amount of consumed coffee is $n - x$. Kate drank one cup of coffee with milk consisting of one fourth of all the milk ($\frac{x}{4}$ of a cup) and one sixth of all the coffee ($\frac{n-x}{6}$ of a cup). Hence $\frac{x}{4} + \frac{n-x}{6} = 1$, i.e., $x = 12 - 2n$. Since $x > 0$, one has $n < 6$; since $x < n$, one has $12 - 2n < n$, so $n > 4$. Hence, $n = 5$.

7.44. Suppose the required number is x. Then $x + 16 = m^2$ and $x - 16 = n^2$, hence $m^2 - n^2 = 32$. The product $(m - n)(m + n) = m^2 - n^2$ is a decomposition of the number 32 into factors with $m - n < m + n$. Such decompositions are the following: $1 \cdot 32$, $2 \cdot 16$, and $4 \cdot 8$. In the first case, $m = \frac{33}{2}$ and $n = \frac{31}{2}$, but these numbers are not integers. In the second case, $m = 9$, $n = 7$, and $x = 65$. In the third case, $m = 6$, $n = 2$ and $x = 20$.

7.45. The digits x and y appearing in the decimal expressions for these numbers both differ from 5; besides, they are odd and not identical. Let us suppose that $x > y$. Then $10x + y - 10y - x = 9(x - y)$ is an exact square, hence $x - y = 4$. Therefore, $x = 7$ and $y = 3$.

7.46. Suppose that x, y, and z snowballs hit Timmy, Johnny, and Billy, respectively. Each of them did throw snowballs after being hit, so the numbers x, y, and z are positive. On the one hand, $13 + x + y + z$ snowballs were thrown. On the other hand, Timmy threw $6x$ snowballs, Billy threw $5y$, Johnny threw $4z + 1$ (counting the first one). Thus we have the equation

$$6x + 5y + 4z + 1 = 13 + x + y + z,$$

i.e., $5x + 4y + 3z = 12$. In particular, $x \leq 2$ and $z < 4$; rewriting the equation in the form $4(x + y + z) + (x - z) = 12$, we see that $x - z$ is divisible by 4. Having in mind the restrictions on x and y, we obtain $x = z$. Hence $8x + 4y = 12$, i.e., $2x + y = 3$, which implies $x = y = 1$, because $y > 0$.

7.47. Suppose the boy had lived x years and y months by the mentioned date. Then $12x + y - x = 111$, i.e., $11(10 - x) + 1 = y$. Since $y < 12$, we have $x = 10$ and $y = 1$.

7.48. For instance, Billy can take one large pill, plus one midsize pill, plus 73 small pills. There are seven other possibilities:

$$(1, 2, 63),\ (1, 3, 53),\ (1, 4, 43),\ (1, 5, 33),\ (1, 6, 23),\ (1, 7, 13),\ (1, 8, 3).$$

Indeed, having swallowed one pill of each type, he will have swallowed $11+1.1+0.11 = 12.21$ grams of antimatter. Additionally, he is to swallow 7.92 grams of antimatter. Then he cannot take any more large pills, but he *can* swallow any number of midsize pills from zero to seven.

7.49. The conditions imply that the number of calves is divisible by 10. Suppose there were a bulls, b cows, and $10c$ calves. For these numbers we obtain two equations $a + b + 10c = 100$ and $20a + 10b + 10c = 200$ (i.e., $2a + b + c = 20$). The first equation implies that $a + b$ is divisible by 10, while the second, that $a + b < 20$. Hence $a + b = 10$, $c = 9$, and $2a + b = 11$. Therefore, $a = 1$, $b = 9$, and $c = 9$.

7.50. Assume that $x^2 - y^2 = (x - y)(x + y) = 14$. The numbers $x - y$ and $x + y$ have the same parity, hence their product is either odd or divisible by 4. We arrived at a contradiction.

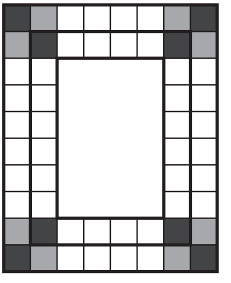
7.51. Suppose the side of the square contains n musicians, and after their rearrangement the number of musicians in a file decreased by x. Then $n^2 = (n + 5)(n - x)$, i.e., $n = \dfrac{5x}{5 - x}$. The only natural number x for which n is a natural number, is 4.

7.52. Write the equation in the form $(x - 5)(y - 2) = 11$. The number 11 can be decomposed into two integer factors in four different ways, yielding four solutions.

7.53. *First solution.* The problem reduces to solving the equation $2(x+y) = xy$ in natural numbers, which can be done similarly to Problem 7.52: the equation can be rewritten as $(x-2)(y-2) = 4$.

Second solution. Obviously, the length and the width of the rectangle is greater than 1. The value of the area of the boundary frame of width 1 is less by 4 than the perimeter. Therefore, the area of the rectangle inside the frame is 4. There are two such rectangles: 2×2 and 1×4. Hence, the initial rectangle has the dimensions 4×4 or 3×6.

7.54. *First solution.* If the dimensions of the rectangle are $a \times b$, then the dimensions of the frame are $(a+2) \times (b+2)$, hence $(a+2)(b+2) = 2ab$, so $2a + 2b + 4 = ab$, i.e. $(a-2)(b-2) = 8$. So either one of the numbers $a-2$ or $b-2$ equals 1, while the other equal 8, or one of the numbers $a-2$ or $b-2$ equals 2, while the other equals 4.



Second solution. It is clear that the width of the rectangle is more than one square. Let us draw inside the rectangle one more frame of width one (see the figure). Then in the smaller frame, as well as in the bigger one, there are four corner squares (they are shown in black), and each side will be shorter by two squares. Then in the smaller frame, there will be 8 squares less than in the larger one (these eight squares are shown in grey). Hence, the rectangle limited by the smaller frame consists of 8 squares. A rectangle of 8 squares must have the dimensions 2×4 or 1×4. This leads us to the answer.

7.55. The identity $(10a+b)^2 = 100a^2 + 20ab + b^2$ implies that the next-to-last digit of the square of the number $10a+b$ is odd only if the square of the tens digit of the number b^2 is odd. Only the squares $4^2 = 16$ and $6^2 = 36$ possess this property.

7.56. Suppose the initial number is $700+10x+y$. Then $700+10x+y = 100x+10y+7+117$, i.e., $90x+9y = 576$. After dividing by 9, we obtain $10x+y = 64$. Hence $x = 6$, $y = 4$, the initial number is 764, the new number is 647.

7.57. The squares of the numbers 44, 45, and 46 are 1936, 2025, and 2116. Accordingly, the nephew could have been born in 1892, 1980, or 2070. In the given context, only 1980 makes sense.

7.58. Denote the number of peaches received by Kolya's, Petya's, Tolya's, and Vassya's sisters by a, b, c, and d respectively. Then $a+b+c+d = 10$

(since the values of a, b, c, and d are 1, 2, 3, and 4 in some order) and

$$(a+b+c+d)+(a+2b+3c+4d)=32,$$

so $b+2c+3d=12$. Therefore, b and d must be of the same parity. If b and d equal 1 and 3, then either $1+2c+9=12$ or $3+2c+3=12$. Both of these cases are impossible. If b and d equal 2 and 4, then either $2+2c+12=12$, or $4+2c+6=12$. The first of these cases is impossible, in the second one, we have $b=4$, $d=2$ and $c=1$, so that $a=3$. Hence Kolya's sister got 3 peaches, Petya's sister, 4 peaches, Tolya's sister, 1 peach, while Vassya's sister got 2.

7.59. If all the thirty jumps were by 7 metres, the kangaroo would have covered 10 meters more than necessary. The replacement of one 7-metre jump by a 5-metre one decreases this distance by 2 m, while its replacement by a 3 m one, decreases it by 4 m. The number 10 can be represented as the sum of twos and fours as 2+2+2+2+2, or 2+2+2+2, or $2+4+4$.

7.60. Suppose $p=x^2-y^2=(x-y)(x+y)$. Since p is prime and $x-y<x+y$, we have $x-y=1$ and $x+y=p$, so $x=\dfrac{p+1}{2}$ and $y=\dfrac{p-1}{2}$.

7.61. It is clear that x^2 is divisible by 5, hence by 25, and is less than $115=345/3$. Hence there are only three possibilities: $x^2=0$, or $x^2=25$, or $x^2=100$. Accordingly, $y^2=69$, or $y^2=54$, or $y^2=9$. Only he third version works.

7.62. Let us decompose the left-hand side into factors:

$$(1+x)(1+x^2)=2^y.$$

Clearly, both factors in the left-hand side must be powers of two. If x is even, then the left-hand side must be odd, which is impossible. If x is odd, then the number $1+x^2$ has remainder 2 under division by 4, hence $1+x^2=2$ and $x=1$.

7.63. The right-hand side is divisible by 7, hence one of the numbers is 7. Up to a permutation, we can assume that it is r. The problem reduces to solving the equation $p+q+7=pq$, which can be written in the form $(p-1)(q-1)=8$. Since none of the factors can equal 8 (in that case the corresponding prime number would equal 9), we have $p-1=2$, $q-1=4$ (or vice versa).

7.64. Under division by 3, the number 2^n has remainder 1 if n is even and the remainder 2 if it is odd. If n is odd, then 2^n+33 has remainder 2 under division by 3, so it cannot be a square (Problem 4.97). Suppose $n = 2k$ and $2^n + 33 = m^2$. Then $(m - 2^k)(m + 2^k) = 33$. This implies $m - 2^k = 1$, $m + 2^k = 33$ or $m - 2^k = 3$, $m + 2^k = 11$. In the first case $2^k = 16$, in the second case, $2^k = 4$.

7.65. We are to find a natural number $n > 9$ such that

$$(n-9)^2 + \ldots + (n-1)^2 + n^2 = (n+1)^2 + \ldots + (n+9)^2.$$

Therefore,

$$\begin{aligned} n^2 &= (n+9)^2 - (n-9)^2 + \ldots + (n+1)^2 - (n-1)^2 = \\ &= 2n \cdot (18 + 16 + \ldots + 2) = 180n. \end{aligned}$$

Hence $n = 180$ and $171^2 \cdots + 180^2 = 181^2 \cdots + 189^2$.

7.66. Assume that $4^k - 4^l = 10^n$. Then

$$k > l \geq 0 \quad \text{and} \quad 4^l(4^{k-l} - 1) = 10^n,$$

where $k - l > 0$. Under division by 3, any power of four has the remainder 1, hence the left-hand side is divisible by 3, whereas the right-hand side is not.

7.67. Let us write the equation in the form

$$x^3 = (2y+1)^2 - 4 = (2y-1)(2y+3).$$

The odd numbers $2y - 1$ and $2y + 3$ are coprime since their difference is 4. Hence both are cubes, but the difference of two cubes of odd numbers cannot equal 4. The latter assertion can be easily proved by using the identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

7.68. Let us prove that $(0, 0, 0)$ is the only integer solution. To that end, assume that there is a solution (x, y, z) in which at least one of the numbers x, y, and z is nonzero. If these numbers have a nonzero common divisor $t > 1$, then $\left(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}\right)$ is also a solution. Hence we can assume that the numbers x, y, and z have no common divisor. But the equation has no such solutions. Indeed, the right-hand side is divisible by 4; hence, it is even, so the left-hand side must have an even number of odd summands. Moreover, there must exist at least one odd summand (otherwise, x, y, and z would have 2 as a common divisor). Hence precisely two of the numbers x, y, z are odd. But the sum of the squares of two odd and one even summand is not divisible by 4, since the square of an odd number has remainder 1 under division by 4 (Problem 4.88). Therefore the equation has no nonzero solutions.

7.69. Suppose that n is even, $n = 2k$. Then the equation can be written in the form

$$(m - 3^k)(m + 3^k) = 55.$$

Hence $3^k < m + 3^k \leq 55$, i.e., $k \leq 3$. Substituting the values $n = 2, 4, 6$, we obtain respectively $m^2 = 55 + 9 = 8^2$, $m^2 = 55 + 81 = 136$ (not a square), $m^2 = 55 + 729 = 28^2$.

Now assume that n is odd, $n = 2k + 1$. Then the the equation can be written in the form $3 \cdot 9^k + 55 = m^2$. The number 9^k ends either in 9 or in 1, hence the last digit of the left-hand side is either 2 or 8, but a square cannot end in such a digit (a square can only end in 0, 1, 4, 5, 6, or 9).

7.70. Let us try to look for an example in which one number has just one digit before and one digit after decimal point, while the other is a whole one-digit number. One obtains the equation $\left(a + \frac{b}{10}\right)c = a + b + c$. If one puts $c = 5$, then $8a = b + 10$. This equation has the solution $a = 2$, $b = 6$.

7.71. Clearly these numbers cannot be one-digit. Nor can they be two-digit, because if

$$10a + b = 13(a + b),$$

then $3a + 12b = 0$. For three-digit numbers, we obtain the equation

$$100a + 10b + c = 13(a + b + c),$$

i.e., $29a = b + 4c$. Clearly $a = 1$, and so we find the solutions indicated in the answers section. One can show that the sum of digits of a number with four or more digits is more than 13 times less than the number itself.

7.72. The number of books received at the first store is divisible by $37 \cdot 11 \cdot 2 = 814$, at the second store, by $\text{LCM}(19, 9, 3) = 171$, at the third one, by $\text{LCM}(25, 30, 10) = 150$. Suppose the first store received $814x$ books, the second, $171y$, and the third, $150z$. Then $814x + 171y + 150z = 1990$. The number y is even, hence $814x \leq 1990 - 2 \cdot 171 - 150 = 1498$. Thus, $x = 1$, and we arrive at the equation $171y + 150z = 1176$. The last digit of the number y is 6 and $y < 16$, hence $y = 6$, $171y = 1026$, and $z = 1$.

7.73. Suppose that in the family there are m boys and n girls; since every child has a brother, $m > 1$. The sum of the numbers in the answers the siblings have given equals $mn + (m-1)m = m(n+m-1)$. Since this sum equals 35 and $m > 1$, the number m may be equal only to 5, 7, or 35. The equation $n = \frac{35 - m^2 + m}{m} = \frac{35}{m} - m + 1$ shows that $n < 0$ for $m = 7$ or 35. Hence $m = 5$ and $n = 3$.

7.74. If one of the numbers m or n equals 1, then the the other also equals 1. In what follows, we assume that both numbers are greater than 1. The collection of prime divisors of the numbers m or n is the same. Besides, $(nm)^{n+m} = n^{12}m^3 < (nm)^{12}$, hence $n + m < 12$. Now a simple case by case inspection shows that either $(n, m) = (2, 4)$ or $(n, m) = (4, 2)$. Of these two pairs, only the first one satisfies the equation.

Chapter 8. Inequalities

8.1. *First solution.* The first number equals

$$(1-2) + (3-4) + (5-6) + \ldots + (99-100) = -50,$$

since the difference between each pair of numbers in brackets equals -1 and there are 50 such differences. Similarly, the second number equals

$$(1+2) + (-3+4) + \ldots + (-99+100) = 3 + 49 = 52.$$

Hence the second number is greater than the first.

Second solution. Bringing together the summands as above, we see that the first number is negative (it is a sum of negative numbers). The sum of the two given numbers equals

$$(1-2+3-4+\ldots+99-100)+(1+2-3+4-\ldots-99+100) = 1+1 = 2$$

(the other summands cancel out). Since the sum of the two numbers is positive, while the first number is negative, it follows that the second number is positive, and therefore greater than the first.

8.2. Suppose the numerator of the required fraction is n. Then $\frac{11}{17} < \frac{n}{16} < \frac{12}{17}$, i.e.,

$$11\left(1 - \frac{1}{17}\right) < n < 12\left(1 - \frac{1}{17}\right), \quad \text{or} \quad 11 - \frac{11}{17} < n < 12 - \frac{12}{17}.$$

Only the natural number $n = 11$ satisfies these inequalities.

8.3. Suppose the numerator of the required fraction is n. Then $\frac{8}{25} < \frac{n}{26} < \frac{9}{25}$, so

$$8\left(1+\frac{1}{25}\right) < n < 9\left(1+\frac{1}{25}\right) \quad \text{or} \quad 8+\frac{8}{25} < n < 9+\frac{9}{25}.$$

Only the natural number $n = 9$ satisfies these inequalities.

8.4. If the salary is first decreased by $a\,\%$, and then increased by $a\,\%$, then it will decrease, because for any non-zero number x we have the inequality

$$(1-x)(1+x) = 1 - x^2 < 1.$$

For it to return to the initial value, we must increase it by more than $a\,\%$.

8.5. It follows from what we know that the total number of eaten cakes is divisible by 2 and 7. The Tin Man and the Wizard together ate one seventh of all the cakes, i.e., an even number which is at most 4. On the other hand, the Wizard ate at least one cake, so the Tin Man, who ate more than the Wizard, ate more than one. Hence the Wizard ate only one cake, the Tin Man ate three, and a total of $4 \cdot 7 = 28$ cakes was eaten. Of them the Scarecrow and the Tin Man together ate one half, i.e., 14, so the Scarecrow ate 11 cakes. Similarly, the Lion and the Wizard together ate 14 cakes, so the Lion ate 13 cakes.

8.6. The number of girls present is divisible by 2 and by 3, hence by 6, while the number of boys is divisible by 4. Denote these numbers by $6d$ and $4m$ respectively. The number of members in each team is the same, hence $2m+2d = 3d+m$, so $m = d$. Since $6d+4m < 27$, this implies that $10d < 27$, hence $m = d = 1$ or $m = d = 2$. One sixth of the girls and one fourth of the boys helped the umpire, hence the number of helpers was $m + d$. Thus there were either two or four helpers.

8.7. Let us number the musketeers in decreasing order of strength: 1 is the strongest one, 4 is the weakest. They can be divided into pairs in the following ways: $(1,2)$–$(3,4)$, $(1,3)$–$(2,4)$, $(1,4)$–$(2,3)$. In the first two cases the first pair wins (in the first case, easily, in the second, not easily), in the third, the result can be arbitrary (in the described scenario, it was a draw). Therefore, the pair $(1,2)$ is Porthos with d'Artagnan, the pair $(1,3)$, Porthos with Athos. Hence 1 is Porthos, 2 is d'Artagnan, and 3 is Athos.

8.8. The equation $a+b = c+d$ together with the inequality $b+d > a+c$ imply

$$a+b+b+d > c+d+a+c,$$

so $2b > 2c$ and $b > c$. Therefore, $d - a = b - c > 0$, so $d > a$. The number c is positive, hence $a > b + c > b$, so $a > b$.

8.9. The group of children for whom the number of flags in both hands becomes the same after the shifting from hand to hand includes one third of all the children, while the group of those who had the same number of flags in their hands before the shifting includes one sixth of all the children. Nobody belongs to both these groups, since if somebody has the same number of flags in both hands, after the shifting these numbers will differ. Each child in each of these two groups has an even number of flags, hence no less than $1/3 + 1/6 = 1/2$ hold an even number of flags. And for each of those who have one flag more in one hand than in the other, the number of flags is odd. So the number of these children is at most one half the total number of children.

8.10. Suppose that such numbers exist. Obviously, x is the smallest, hence it is one-digit. Therefore,

$$1/y + 1/z \leq 1/10 + 1/100 = 0.11 < 1/9,$$

so $x > 9$, a contradiction.

8.11. We can assume that $a \leq b \leq c$. Consider three cases.

1. Suppose that $a = 2$. Then $b > 2$.

If $b = 3$, then $c > 6$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42}$.

If $b = 4$, then $c > 4$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{5} < \frac{41}{42}$.

If $b > 4$, then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{2} + \frac{1}{5} + \frac{1}{5} < \frac{41}{42}$.

2. Suppose that $a = 3$.

If $b = 3$, then $c > 3$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{4} < \frac{41}{42}$.

If $b \geq 4$, then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{4} < \frac{41}{42}$.

3. Suppose that $a > 3$. Then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} < \frac{41}{42}$.

8.12. Weights of mass 49.5 g, 50.5 g, and 51.5 g will do.

8.13. None of the weighings was performed by using a single weight, else its mass would be greater than 90 g. Suppose that in some weighing three weights were used. This could have been only the third weighing, so the sum of the three weights equals 102 g. Then, in order to get 100 g and 101 g, one of the weights must have been 1 g, and another, 2 g. Hence the third weight must have been 99 g, which doesn't fit us. Therefore, in all the weighings, exactly two weights were used, and these three pairs were different. Therefore, the doubled mass of all the weights equals $100 + 101 + 102 = 303$ g, i.e., the sum of masses of the three weights is 151.5 g. Hence the weight of the lightest one is $151.5 - 102 = 49.5$ g, of the next one, $151.5 - 101 = 50.5$ g, and that of the heaviest one, $151.5 - 100 = 51.5$ g.

8.14. Suppose that in the initial project there were 5 entrances, two floors, and one flat on each floor, 10 flats in all. Then after the project was first modified, there were 3 entrances, 5 floors, and 15 flats. After the second modification, there were 1 entrance, 8 floors, and 8 flats.

8.15. If there are no more than three flats on each floor, then there are no more than $10 \cdot 9 \cdot 3 = 270$ flats in the first ten entrances, so there is no flat with number 333 in the tenth entrance. It there are 5 or more flats on each floor, then there are at least $9 \cdot 9 \cdot 5 = 405$ flats in the first nine entrances, and no flat number 333 in the tenth entrance either. Thus there are 4 flats on each floor and $9 \cdot 9 \cdot 4 = 324$ flats in the first nine entrances, the number of the first flat in the tenth entrance is 325 and the flat number 333 is on the third floor.

8.16. Since $2^3 < 3^2$, one has $(2^3)^{100} < (3^2)^{100}$, i.e., $2^{300} < 3^{200}$.

8.17. One has $31^{11} < 32^{11} = 2^{55} < 2^{56} = 16^{14} < 17^{14}$.

8.18. If we subtract the second fraction from the first, then we obtain a fraction whose numerator equals

$$(10^{10} + 1)(10^{12} + 1) - (10^{11} + 1)^2 = 10^{10}(101 - 20).$$

8.19. The numbers p and q cannot be equal, hence we can assume that $p < q$.

First solution. The number $7p + 7q + 1$ is divisible by pq, hence

$$7p + 7q + 1 \geq pq.$$

If $p > 7$, then

$$(p - 7)^2 < (p - 7)(q - 7) = (pq - 7q - 7p - 1) + 50 \leq 50,$$

i.e., $p \leq 14$. Hence p can only take the values 2, 3, 5, 7, 11, or 13. Then $7p + 1$ accordingly takes the values 15, 22, 36, 50, 78, or 92. A simple verification shows that only the three pairs given in the answer section meet the requirements of the problem.

Second solution. If $p = 2$, then q divides 15, so $q = 3$ or $q = 5$, and both variants satisfy the conditions. Suppose that $p \geq 3$. Then the numbers $7p+1$ and $7q+1$ are even: $7p+1 = 2aq$ and $7q+1 = 2bp$. Clearly, $2ap < 2aq < 2bp$, hence $a < b$. Further,

$$4ab = 2a \cdot 2b = \frac{7p+1}{q} \cdot \frac{7q+1}{p} = 49 + \frac{7}{p} + \frac{7}{q} + \frac{1}{pq}.$$

Therefore,

$$49 < 4ab \le 49 + \frac{7}{3} + \frac{7}{5} + \frac{1}{15} < 53,$$

and so $4ab = 52$, i.e., $ab = 13$. Thus, $a = 1$, $b = 13$, $7p+1 = 2q$, and $7q+1 = 26p$. Therefore, $49p + 7 = 14q = 52p - 2$, hence $p = 9 : 3 = 3$ and $q = (7 \cdot 3 + 1) : 2 = 11$.

8.20. If the first digit is 0 or 2, then the sum of the first two digits is less than $1 + 9$. If the third digit is not 5, then the sum of the last two digits is less than $5 + 9$.

8.21. Suppose s pikes are satiated. Then all together they have eaten at least $3s$ pikes. A pike cannot be eaten twice, and at least one pike remained at the end, hence $3s < 30$ and $s \le 9$. Let us give an example with 9 satiated pike. Suppose 7 pikes (from the third to the ninth) ate up 21 pikes (from the tenth to the thirtieth; each ate 3 pikes). After that 9 pikes remain. The first and second can be satiated by eating up 6 pikes (from the fourth to the ninth).

8.22. If both numbers a and b are greater than 1, then $(a-1)(b-1) > 0$, hence $ab+1 > a+b$. Hence, if among the billion numbers whose product is one billion there are two numbers a and b greater than 1, their sum will increase and their product will remain the same if they are replaced by ab and 1. Hence the sum is maximal when one number is 10^9 and the others are equal to 1.

8.23. Observe that

$$(a+d)(b+c) - (a+c)(b+d) = (a-b)(c-d) > 0.$$

8.24. The greater the first digit, the better, so the first digits must be 7, 8, and 9. This applies to the second digits as well, but, according to Problem 8.23, when distributing these second digits among the three numbers, the smaller second digit must follow the greater first digit. The same argument applies when distributing the remaining digits 1, 2, and 3 as third digits.

8.25. See the answers section.

8.26. The bunnies can't drum all at once, because the one with the smallest drum won't be able to use it. On the other hand, if that bunny is given the shortest pair of sticks, all the others will start drumming.

8.27. When 4 guests have left, for at most 4 of the remaining 6 guests the galoshes could have been taken away, so at least one of these six guests can put on their galoshes and leave. If the first to leave are the 5 guests who came in the smallest galoshes and if they put on the largest ones, then the remaining 5 guests will not be able to put on any galoshes.

8.28. Suppose that x people attended the meeting. On the one hand, exactly 48 citizens are dissatisfied by each of the reforms, so there is $48 \cdot 5 = 240$ dissatisfactions. On the other hand, each participant in the meeting is dissatisfied with at least three reforms, hence the total number of dissatisfactions is no less than $3x$. Thus, $240 \geq 3x$, so that $x \leq 80$.

Let us give an example when exactly 80 people attend the meeting. Choose 80 inhabitants of the island and divide them into five groups of 16 people (and assume that the remaining 16 are quite happy with all the reforms). Suppose people from the groups 1, 2, and 3 object to the first reform, people from groups 2, 3, and 4, to the second, people from groups 3, 4, and 5, to the third, people from the groups 4, 5, and 1, to the fourth, groups 5, 1, and 2 to the fifth. Then exactly $3 \cdot 16 = 48$ people object to each reform, and 80 citizens will come to the meeting.

8.29. Let us decrease the required number by 56. As a result, we obtain a number divisible by 56, with sum of digits 45, ending in 00. Since the number $56 = 7 \cdot 8$ is divisible by 8, the third digit from the end must be even. The least number with sum of digits 45 ending in 00 is 19 999 800. But it is not divisible by 7. The next such number is 28 999 800, and it not divisible by 7 either. The third such number, 29 899 800, is already divisible by 7.

8.30. Suppose the clock shows the time $ab{:}cd$, where a, b, c, and d are different digits. If $a = 0$, then more than one hour must pass until four new different digits are lighted up (10:?? does not work, because the digit 0 appeared before). If $a = 2$, then again more than one hour must pass until four new different digits are lighted up (the next possible version is only 01:??). If $a = 1$, then we must look for an interval from $19 : xy$ to $20 : zt$. It is easy to verify that the shortest such interval is from 19:58 to 20:34.

8.31. For x kilograms of gold and y kilograms of diamonds, Aladdin will get $20x+60y$ dinars. He cannot lift more than 100 kg, hence $x+y \leq 100$. Besides, 1 kg of gold takes up $\frac{1}{200}$ of the chest, while 1 kg of diamonds takes up $\frac{1}{40}$. Hence, $\frac{x}{200} + \frac{y}{40} \leq 1$, i.e., $x + 5y \leq 200$. Adding the inequalities $x + y \leq 100$ and $x + 5y \leq 200$, we obtain $2x + 6y \leq 300$. Therefore, $20x + 60y \leq 3000$, and so Aladdin can make no more than 3000 dinars from the treasures from the cave. Solving a system of linear equations, we find that $x + y = 100$ and $x + 5y = 200$ if $x = 75$ and $y = 25$. This means that Aladdin can make 3000 dinars by taking 75 kg of gold and 25 kg of diamonds.

8.32. Let us put the tangerines aside for the moment. There will remain $20 + 30 + 40 = 90$ fruits. Each happy monkey must have eaten at least two of the remaining fruits, so there are at most $90 : 2 = 45$ happy monkeys. Let us show that one can make 45 monkeys happy: 5 monkeys eat a pear, a banana, and a tangerine, 15 monkeys, a pear, a peach, and a tangerine, 25 monkeys, a peach, a banana, and a tangerine.

8.33. Three painters and 6 fitters can do the work in 195 minutes. Indeed, if each painter works in a team with two fitters, then in 35 minutes one painter and the first fitter will produce the first item, and 10 minutes later, the second fitter will finish putting together the second item. The third item will finish drying exactly at the moment when the first fitter will be free, and 10 minutes after that, the third item will be ready. Further on, a new item will be ready every 10 minutes. Thus, in $35+16\cdot 10 = 195$ minutes, one painter and two fitters will produce 17 items, while 3 painters and 6 fitters will produce 50 items (even $3 \cdot 17 = 51$ items).

If there are less than three painters, the time needed for painting will total no less than 250 minutes, while if there are less than six fitters, the time needed for putting the items together will be no less than 200 minutes. Hence there must be 3 painters and 6 fitters, to which we can add one painter or one fitter. The addition of one fitter will not speed up the conclusion of the process: the painters work nonstop, while the 6 fitters begin putting the items together immediately, as soon as it dries. The addition of one painter speeds up the process at the very beginning, but will not result in the whole process ending sooner. When the second items painted by the four painters have dried, the fitters won't be able to start putting them together immediately. And if the fitters will make pauses in their work, the whole process will take longer.

8.34. We can delete the first digits 1234, leaving 5 as the first digit;

it will be the largest possible first digit. After that, we can delete the digits 1234 that follow the remaining 5, leaving the two digits 55. We have deleted 8 digits, it remains to delete two. The greatest number will be obtained if we delete the two digits 12 standing after the first two digits 55.

8.35. The number under consideration consists of $9+2\cdot 51 = 111$ digits. Hence, after the deletion, 11 digits will remain.

a) There is a total of 6 zeros in the number, but only 5 of them can be leading zeroes. Deleting all the other digits before them, we obtain the number 00000515253 . . . 5960. Besides the first five zeros, we must leave 6 digits. We can leave the digits 123450. Then all the deleted digits are greater than 4, hence the number 123450 is the smallest.

b) As in the solution of item a), we first obtain the number

$$100000515253\ldots5960.$$

Besides the first 6 digits, we must leave 5 digits. We can leave the digits 12340. Then all the deleted digits are greater than 4, hence the number 12340 is the smallest.

c) The number contains 6 nines. We can leave only 5 of them at the beginning. Deleting all the other digits before these 5 nines, we obtain the digits 999995051 . . . 57585960. Besides the first 5 nines, we must leave 6 digits. In the number 5051 . . . 57585960, the greatest digit after which there remain no less than 5 digits, is 7. In the number 585960, we must delete one digit. Clearly, we must delete the first one in order to obtain the greatest number.

8.36. The first 10 prime numbers form the sixteen-digit number

$$2\,357\,111\,317\,192\,329.$$

The largest digit that can remain after deleting 6 digits is 7. In the number 7111317192329 we must delete 3 digits so that the greatest number will remain. Clearly, we must delete the first three 1's after the seven (i.e., the group 111).

8.37. It is said that the rightmost is the Tin Man. In particular, he sits further to the right than the Scarecrow. After the Tin Man changed his position, there was no one further to the left than the Scarecrow. Hence there was none there initially. i.e., the leftmost is the Scarecrow, and next to him is Dorothy. Hence, the Lion sits to the right of her.

8.38. Let us rank the people in increasing order of gathered mushrooms: $a_1 < a_2 < a_3 < \ldots < a_{10}$. Clearly, $a_1 \geq 0$, $a_2 \geq 1$, $\ldots$, $a_{10} \geq$

9. Let us take away all the mushrooms gathered by the first person, leave 1 mushroom to the second person, 2 to the third, ..., 9 to the tenth. Then the mushroom pickers will still have $0 + 1 + 2 + \ldots + 9 = 45$ mushrooms in all. Hence one mushroom was taken away from them. It could have been taken only from the person ranked tenth, because otherwise two persons would have gathered the same number of mushrooms.

8.39. The tallest of the girls is Lusya Egorov. She must have skated with a boy taller than she is. There are two such boys, but one of them is her brother. Hence, Lusya skated with Yura Vorobiev. Similarly we find out that Olya skated with Andrey, Inna with Seriozha, Ania with Dima.

8.40. Let A be the shortest among the tallest ones, B the tallest among the shortest ones, and let C be the person at the intersection of the file containing A and the rank containing B. Then A is not shorter than C and C is not shorter than B, so A is not shorter than B.

8.41. Yesterday, "tomorrow" was today, while the day after tomorrow, "yesterday" is tomorrow. Hence, if "tomorrow for yesterday" was Thursday, then "yesterday for the day after tomorrow" is Friday.

8.42. "The puzzle that you solved after you solved the puzzle before you solved that one" is the same as "that one".

8.43. Let A be the lightest weight, and let B be the next lightest weight after A. The pair of weights (A, B) can only be in balance with the same kind of pair. Hence there must be at least two weights of mass A and at least two weights of mass B. The pair (A, A) can also be in balance only with the same kind of pair. Hence there are at least 4 weights of mass A. For similar reasons, there are 4 weights of the maximal mass E and at least two weights of the next to maximal mass D. Besides, since there are weights of at least five different masses, there is a weight of mass greater than A and B and less than D and E. So there are at least $4 + 4 + 2 + 2 + 1 = 13$ weights. It is easy to check that the collection of weights

$$1, 1, 1, 1, 2, 2, 3, 4, 4, 5, 5, 5, 5$$

meets all the conditions.

8.44. Assume that after the rearrangement the fifth form pupil Johnny is taller than the sixth form pupil Freddy, who stands behind Johnny. We can assume that the height in the rows increases from left to right.

Suppose that in Johnny's row there are n people to the left of Johnny and m to his right. Johnny and these m fifth form pupils to the right of him are all taller than Freddy and each of the n sixth form pupils to Freddy's left. Hence, at most n fifth form pupils may be shorter that Freddy or any sixth form pupil to Freddy's left. However, originally for each of these $n+1$ sixth form pupils there was a shorter fifth form pupil. We arrived at a contradiction.

Chapter 9. Logic, Combinatorics, Sets

9.1. The third sack contains flour, hence the first sack contains sugar.

9.2. The inscription on the blue can implies that there is something in the green can. The inscription on the green can implies that it is not wheat.

9.3. Cinderella took a grain from the sack tagged "mixture". That sack could not contain the mixture, so Cinderella immediately understood what its contents were. Suppose, say, that it was a seed of poppy. Then the sack tagged "poppyseed" contains only wheat. Indeed, had it contained the mixture, then the sack with the tag "wheat" would contain wheat indeed, which is impossible. So the sack tagged "wheat" must contain the mixture. If the grain Cinderella took was a grain of wheat, the argument is similar. Try and reproduce it.

9.4. If the Tin Man was right, then so was the Lion, but one of them was wrong. So the Tin Man was wrong and the Lion was right. Hence Dorothy is more than 10 years old, but not more than 11. Therefore she is 11.

9.5. *First solution.* If the dragon was killed by Muromets, then all three lied, which contradicts the condition. If it was Nikitich, then Muromets told the truth, and Popovich and Nikitich lied; this agrees with the condition. If it was Popovich, then both Nikitich and Popovich told the truth, and this also contradicts the condition. So only Nikitich could have killed the dragon.

Second solution. Nikitich and Popovich assert the same thing. But only one strong man told the truth, so they both lied, so it was Muromets who told the truth, so it was Nikitich who killed the dragon.

9.6. If Harry stole the rum, then he told the truth once and lied once, while in that case Tom lied twice and Charley told the truth twice. This agrees with the condition. If Tom stole the rum, both Harry and Tom told the truth once and then lied the second time, which contradicts the condition. Now if Charley is the thief, then both Harry and Tom told the truth twice, and this also contradicts the condition.

9.7. If Andrey told the truth, then Dima lied, so at least one of them lied. Three brothers cannot have lied, so in this case Nikita or Gleb told the truth, which means that Igor lied. Hence Nikita and Gleb both told the truth. Therefore, Igor baked the pie. (This agrees with the condition: Nikita, Gleb, and Dima told the truth, Igor and Andrey lied.)

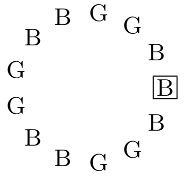
9.8. If John was right, then both girls were mistaken, because the number 9 is odd and is not divisible by 15. Hence, of the two boys it was Peter who answered correctly. But no prime number is divisible by 15, while the only even prime number is 2.

9.9. Suppose Petya has n books. If $0 < n < 10$, then the second and third assertions are true, while if $n > 10$, then the first and third ones are true.

9.10. Suppose that an inhabitant of A telephoned. Then the first statement implies that the fire is in A, while the second, that it is in C. This is impossible. Suppose that the call came from an inhabitant of C. Then his statements are either both true (if the fire is in C) or both false (if the fire is not in C). This is also impossible. Hence an inhabitant of B actually called. But he lied twice, so the fire is actually in A.

9.11. If the first ladybug tells the truth, then the second and third must also tell the truth, since they must have as many dots on their backs as the first. But the second and third ladybugs contradict each other, hence at least one of them lies. If each of the first three ladybugs lied, then all the others would also lie, since in that case all the ladybugs are liars. But then what the first ladybug said would be true, which contradicts our assumption. Therefore the first three ladybugs could not have lied simultaneously, so either the second or the third told the truth, while the other two lied. Thus each of the remaining ladybugs said the truth. Therefore, there are two ladybugs that have four dots on their backs, several that have six dots, and the sum of the number of dots of all the ladybugs is either 30 or 26. But the number $30 - 2 \cdot 4 = 22$ is not divisible by 6, so that number is 26, and there are $(26 - 2 \cdot 4) : 6 = 3$ honest ladybugs. Therefore, 5 ladybugs gathered in the clearing.

9.12. It is clear that there are both boys and girls at the table. Following a group of boys sitting together, there is a group of girls, then another group of boys, and so on (a group can consist of one person). The groups of boys and girls alternate, so there is an even number of them. False statements were expressed only when passing from group to group, i.e., there is also an even number of false statements. The statement "most of us are boys" was expressed seven times, hence six statements "most of us are girls" are false, and so there are also 6 groups. The alternation of true and false statements occurs only in groups of 2 children. Only the first and the last child, who sit together, said the same thing, hence there are 3 persons in their group. These persons are boys, because they form the majority. In all there were $2+2+2=6$ girls and $2+2+3=7$ boys. The picture shows how they were actually placed around the table; the first speaker is framed.



9.13. Roma cannot be honest, because in that case Kesha is honest, too, which is impossible. Hence Roma is either the liar or the clever parrot. Suppose Roma is the liar. Then Kesha is the clever one (he can't be the liar, because it's Roma who is the liar, he can't be honest, because he asserts that he isn't). Thus, Gesha is honest, contrary to his assertion. Therefore, Roma is the clever parrot. Then Kesha, judging by his statement, is the liar, and so the only possibility for Gesha is to be honest. His assertion is true indeed. Thus Gesha is honest, Roma is clever, Kesha is a liar.

9.14. Let us consider the creature about whom it was said that he is a hobbit, and call him Bob for convenience. Bob did not agree that he is a hobbit, the next one did not agree with him, and so he confirmed that Bob is a hobbit, and so on — all the speakers alternatively confirm or deny that Bob is a hobbit.

a) If there were 9 (an odd number) feasters, then the next time around each one would say the opposite to what he said the previous time. So they are all hobbits. The first hobbit told the truth about Bob, which is quite possible.

b) The number 10 is even, so all the speakers always repeat the same thing. So there are no hobbits among them. Then Bob isn't a hobbit either, and his neighbour to the right, who said that Bob is a hobbit, was a goblin. Now Bob himself denounced that goblin as a liar, so he is an elf. His neighbour to the left is a goblin again, and so on. Thus, five goblins and five elves were sitting alternatively around the table.

9.15. To the first question "Are you Jackal?" both Lion and Jackal will answer "No". Since Hedgehog found out who is Giraffe, Giraffe answered "Yes" and Parrot answered "No".

To the second question "Are you Giraffe?", Lion and Giraffe will answer "No" (Giraffe was answering the question "Are you Jackal?"). Since after the second round of questions Hedgehog found out who is Jackal, Jackal answered "Yes", while Parrot answered "No".

The answer of the first animal to the third question was sufficient for Hedgehog, while before that answer he didn't have enough information. Hence, neither Giraffe, nor Jackal, nor Lion were the first in line. Indeed, their answers to the question "Are you parrot?" were predictable and cannot bring Hedgehog any new information. So the first in line was Parrot. He answered "Yes", repeating the answer of the fourth animal to the previous question. Thus, the fourth animal is Jackal.

Two variants of the order of the animals remain:

(1) Parrot, Giraffe, Lion, Jackal and

(2) Parrot, Lion, Giraffe, Jackal.

In case (1), the third (Lyon, who said "No") would not have repeated the answer of the second (Giraffe, who said "Yes"). Hence Hedgehog already understood after the first questioning that Parrot is not the third. And so he would know everything after the second questioning.

In case (2), Parrot and Lion both say "No" to both of the two first questions, so after the first two questions it is impossible to distinguish the variants Parrot, Lion, Giraffe, Jackal and Lion, Parrot, Giraffe, Jackal.

9.16. A knight who calls himself knave is lying. A knave who calls himself knave, tells the truth. Neither is possible.

9.17. Both a knight and a knave will call themselves knights. Thus, the guide reproduced the answer correctly, so the guide is a knight.

9.18. See the answers section.

9.19. Both will answer "Yes".

9.20. See the answers section.

9.21. Freddy cannot be a knave since in this case his assertion would be true. Both boys cannot be knights (otherwise Freddy's assertion is false).

9.22. All of them cannot be knaves, since in that case the statement would be true for all. There cannot be more than one knight, since in that case the statement would be false for them. If there is only one knight, then the statement is true for him and false for the others.

9.23. The first inhabitant is a knave (a knight could not have said that there are no knights among them). Hence not all three are knaves (otherwise the first one would have told the truth). Suppose that the second one is a knave. Then the third inhabitant is a knight and the answer given by the second inhabitant is true, which is impossible. Hence the second inhabitant is a knight. He always tells the truth, so the third inhabitant is a knave.

9.24. If the first one is a knight, then his words imply that the second and third are knaves. This contradicts the words of the second one. Hence the first one is a knave. If the second one is a knave, then his words imply that the third one is also a knave. But then the first would have told the truth, while he is a knave. Hence the second is a knight. His words imply that the third is also a knight. Therefore he will truthfully say "One".

9.25. Initially all the knights were sitting on the blue chairs and all the knaves on red ones. Hence the number of knights who moved to red chairs is equal to the number of knaves who moved to blue chairs. Both the knaves that moved to blue chairs and the knights that moved to red chairs said that they are sitting on red chairs. The total number of those claiming that they are sitting on red chairs is 10. Hence there are $10 : 2 = 5$ knights.

9.26. Aladdin can, say, point to one of the caves and ask: "What would you have answered if I asked you yesterday whether there was treasure in this cave?". Independently of whether the guard tells the truth or not on that day, the answer will be false.

9.27. Let us ask one of them the following question: "What would be your brother's answer to the question „Are you Johnny?"?" Johnny, be he a knight or a knave, will say "Yes", Freddy will say "No".

9.28. Had they both given the same answer, Peter would be unable to distinguish them. Hence the second brother answered "No". If Johnny was the first to answer, then it would turn out that both told the truth, which contradicts the statement of the problem. Hence Johnny was the second to answer, and both lied (which agrees with the statement of the problem).

9.29. Two persons who made the same assertion are either both knaves or both knights. In the island, there is at least one knight and at least one knave. So either all the knights made the first assertion, while all the knaves made the second assertion, or *vice versa.* In the first case, there is an even number of knights and an even number of knaves, in the second, an odd number of each one. Hence the number of people in the island is even.

9.30. It is clear that the first person lied, and so the last one told the

truth. Besides, if one of the persons told the truth, the one after him also told the truth. The assertion "There are no more than $k-1$ knights here" is equivalent to the assertion that there is no less than $13-k$ knaves here. If the k-th speaker has not lied, the inequality $k > 13-k$ holds, i.e., $k > 6$. Therefore, there are no more than 6 knights, hence the seventh speaker told the truth and so did the one after him. So no less than 6 persons told the truth, therefore there are exactly 6 knights.
9.31. If there were more than 50 knights, then there was a knight among those who participated in the survey and he will say that knights form a majority. If there were more than 50 knaves, then there will be a knave among those who participated in the survey, and he will say that knights form a majority. So there was an equal number of knights and knaves. In that case both the knights and the knaves would answer that there were more knaves.
9.32. This is possible if the professor is a woman.
9.33. Since among any two balloons one is blue, there cannot be two red balloons in the room. Hence there are 84 blue balloons and 1 red one in the room.
9.34. The brunet person spoke to Whitey, hence they are two different people. Besides, Whitey is not blond. Hence Whitey is a redhead. Therefore, Blacky is blond.
9.35. Neither Assya, nor Katya, nor Nina is wearing the green dress, hence it is Galya who is in green. Katya is not in the green dress, nor in the white one, nor in the pink one, so she is in blue. Thus Galya stands between Katya and Nina, and so Assya also stands between Katya and Nina. The girls stand as follows: Galya (in green), Katya (in blue), Assya, and Nina. From this and from the condition that the girl in white stands between the girl in pink and Katya, it follows that Assya is in white and Nina, in pink.
9.36. Valya's shoes are not pink (by condition) and not red (Masha wears them), so she is in white shoes. The only colour remaining for Nadya's shoes is blue. Her dress is of the same colour, i.e., blue. Valya and Masha wear shoes and dresses of different colours, but not blue, because every thing blue is on Nadya. Hence Valya wears the red dress, while Masha, the white one.
9.37. There is the same number of dogs, cats, and parrots. This number cannot be greater than 1. If there is a dog, a cat, and a parrot, then there are no cockroaches. If there are no dogs, cats, or parrots, then there are two cockroaches.

9.38. Liosha sees two numbers, hence he knows which three numbers remain. If he is sure that the sum of any two of the remaining three is even, then these three numbers must be of the same parity (otherwise the sum of an even and an odd number would be odd). The given collection does not contain three even numbers, so the numbers that Liosha does not see are 1, 3, and 5. So 2 and 4 are written on his cards.

9.39. See the answers section.

9.40. Here any two assertions contradict each other, so they cannot be simultaneously true. Thus, there is at most one true assertion. All the assertions cannot be false, because in that case the last assertion would be true. Thus, exactly one assertion is true, namely the 99-th, which says that 99 assertions are false, and one is true: that assertion itself.

9.41. For convenience, let us number the conditions:

1) Vika stands in front of Sonya, but behind Alla;
2) Borya and Alla are not next to each other;
3) Dennis is not next to Alla, nor to Vika, nor to Borya.

Condition 1 implies that Alla, Vika, and Sonya stand precisely in the indicated order. Neither Dennis nor Borya stand next to Alla (conditions 2 and 3), so Alla is the first, while Vika is the second. Condition 3 implies that Dennis is the last, next to Sonya. Hence Boris is the third.

9.42. For convenience, let us number the conditions:

1) the red figure is between the blue and green one;
2) the rhombus lies to the left of the yellow figure;
3) the disk is to the right of both the triangle and the rhombus;
4) the triangle is not at an extremity;
5) the blue and green figures are not next to each other.

The red figure is between the blue and green one (condition 1), while the yellow one is not next to the blue one (condition 5), hence only two arrangements of the figures according to colour are possible: [blue, red, green, yellow] and [yellow, green, red, blue]. The first configuration contradicts condition 2: to the right of the yellow figure there must be another figure. Hence the order of colours is [yellow, green, red, blue]. Condition 2 implies that the rhombus is green. Therefore condition 4 now implies that the triangle is red. Now condition 3 implies that the disk is blue. Hence the rectangle is yellow.

9.43. Let us verify that the answer given in the answer section does the job. If Muromets was given two gold coins, he will say "Yes", because in that case at most one gold coin remains for Popovich. If both of Muromets' coins are silver, then Popovich was given at least one gold coin and Muromets will answer "No". And if Muromets was given two different coins, he will answer "I don't know", because Popovich can have been given either two silver coins or two gold coins.

One can ask other questions, say, "Is it true that one of the two other strongmen has been given two silver coins?"; "Is it true that the two other strongmen have been given at least one silver coin?"; "If I take away one of your coins and replace it by a gold one, will you then have one gold coin more?" (Note that the last question does not mention the two other strongmen, it deals only with Muromets himself.)

9.44. The logician A cannot have a six (otherwise he would have answered "No") nor an ace (otherwise he would have answered "Yes"). The logician B knows this, hence he cannot have a six, a seven, an ace, or a king. The logician C cannot have a six, a seven, an eight, and ace, a king, or a queen. The logician D cannot have a six, a seven, an eight, a nine, an ace, a king, a queen, or a jack. So he can only have a ten.

9.45. Petya should let Vassya choose his piece.

9.46. One can, for instance, propose the following sharing of the pie. Petya cuts off what he considers his share. If Vassya and Kolya do not object, Petya takes that piece. If Vassya decides that Petya claims too much, he can cut off a piece of the piece cut off by Petya. If after that Kolya objects, then he can cut off a piece of the remaining piece and take the rest. Petya and Vassya are happy because from their point of view Kolya took less than one third of the pie. If Kolya does not object, then Vassya takes what now remains of the piece cut off by Petya. Petya and Kolya are happy: from Petya's point of view, Vassya took less than a third of the pie, from Kolya's point of view, no more than a third. Then the two boys who have not yet taken any pie, can conclude the sharing as in Problem 9.45.

9.47. At the beginning, the first boy cuts off what he considers his share. Then the other boys in order either agree with that division or cut off a piece of the remains of the pie. The last one of them takes the remaining piece of the pie. After that the three boys who have not yet taken any pie divide it as in Problem 9.46.

9.48. Independently of the choice of route used to join A to B, we can go from B to C via three different routes. Hence the number of different routes joining A to C is three times greater than that from A to B, i.e., it equals 6.

9.49. Independently of the choice of envelope, the stamp can be chosen in three different ways. So the required number is three times the number of different envelopes, i.e., it is 6. Note that the argument here is the same as in Problem 9.48.

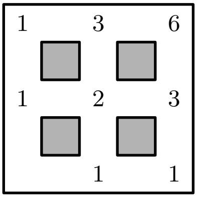
9.50. Independently of our choice of 6 possible routes from A to C, we can go from C to D along 4 different roads. Hence the number of routes from A to D is 4 times that from A to C, i.e., it equals 24.

9.51. From A to C there are 6 routes via B and 2 routes via D. One can travel either via B or via D, not both, so there are $6 + 2 = 8$ routes.

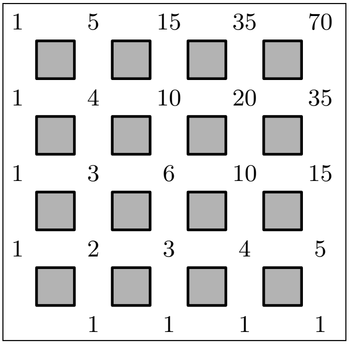
9.52. a) The person who will set up the tent can be any one of the hikers. After that, 9 persons remain, and any one of them can bring water. Thus there are $10 \cdot 9 = 90$ possibilities.

b) First, as in problem a), let us choose the first and second person, obtaining 90 possibilities. But now the first and second persons are doing the same thing. Therefore it is of no consequence which person was chosen first. This means that the number of possibilities must be divided by 2.

9.53. a) At each crossing, let us write the number of ways to get there. This can be easily done: we start from the nearest crossings to A and progressively move on. As a result we will obtain the numbers indicated on the figure to the right.



b) As in problem a), we obtain the numbers indicated on the figure below.



9.54. If two objects are placed in a box, then the objects that will be put in the other box are uniquely determined. The boxes are different, hence for two objects chosen differently a different distribution of objects is obtained. Thus the required number is equal to the number of different choices of 2 objects out of 4 different ones. That number equals $\frac{4 \cdot 3}{2} = 6$.

Indeed, there are 4 ways of choosing the first object, after which there remains 3 ways of choosing the second one. In all we get $4 \cdot 3 = 12$. But it makes no difference which object we choose first or second. So we must divide the above number by 2.

9.55. The pockets are different, hence we can choose one of them and call it the first. Everything is determined by the number of coins in the first pocket, because the second pocket contains all the other coins. The number of coins in the left pocket may be 0, 1, 2, or 3.

9.56. Everything is determined by what coins lie in the first pocket. For each coin there are two possibilities: it is either in the first pocket or it isn't. In all we have $2 \cdot 2 \cdot 2 = 8$ variants.

9.57. Divisibility by 4 depends only on the last two digits.

a) The last two digits can only be 12, 24, and 32. In each of these cases the first two digits can be chosen in two ways. In all we get $2 \cdot 3 = 6$ numbers.

b) The last two digits can only be 12, 24, 32, and 44. In each of these cases any one of the of the first two digits can be chosen in four ways. In all we get $4^2 \cdot 4 = 64$ numbers.

9.58. For each one of the 10 friends there are two possibilities: the person can be invited or not invited. Hence the number of all different variants of choosing 10 friends is $2^{10} = 1024$.

9.59. Suppose first that all the 49 pupils stand in a hall and we successively let them into the classroom so that at any moment they should be divided into the appropriate groups. Suppose the pupil Mary is still in the hall. If she is acquainted with another pupil who is also in the hall, we form a new group of them and let them both into the classroom. If not, then all Mary's acquaintances are already in the class. There are less than 50 pupils, there are at most 24 groups, so among Mary's 25 or more acquaintances, there are two in the same group. If this group consists only of them two, we let Mary in the classroom and add her to the group. If it is a group consisting of three pupils, we choose one of them (with whom Mary is acquainted), ask them to form a group with Mary, and form another group from the two others. Iterating this procedure, we will eventually form the required groups.

9.60. For each MP, let us write down the number of times he/she attended the sessions and add all these numbers. The result is $20 \cdot 5 = 100$. If each MP attended the sessions no more than 4 times, then at least $100 : 4 = 25$ of them attended these sessions, and we are done. Now suppose that one MP attended at least 5 sessions. Then at each of these 5 or more sessions all the MP's present were different, so there were at least $5 \cdot 4 + 1 = 21$ MP's in all.

9.61. a) Let us divide 60 pupils in groups of classmates. If there are no groups of 15 classmates or more, then then there are at most 14 pupils in each group. Let k be the number of groups consisting of two persons or more; we will call these groups "big". One has $k \leq 4$ (otherwise, by taking two pupils from each of the big groups, we will find 10 pupils among which there are no 3 classmates). Now consider two cases.

Case 1. $k \leq 3$. Then the total number of people in the big groups is at most $14 \cdot 3 = 42$. Therefore, 18 pupils are not in any big group, i.e., they have no classmates, which is a contradiction.

Case 2. $k = 4$. Then the total number of people in the big groups is at most 56. Therefore, there are 4 pupils who have no classmates. Taking these four and adding to them two persons from each big group, we will find $12 > 10$ pupils among which there is no group of three classmates, and we arrived at a contradiction again.

b) Suppose there are 4 classes and 15 pupils from each class are present. Then there is no group of 16 classmates, but among any 10 pupils there must be at least 3 classmates (otherwise, in this group there would be at most two pupils from each class and at most $2 \cdot 4 = 8$ pupils in all).

9.62. $15 - 10 = 5$ people studied only mathematics, $21 - 10 = 11$ people, only biology. Hence there were $5 + 11 + 10 = 26$ people in the class.

9.63. *First solution.* The number of people who did not buy strawberry ice cream is $24 - 15 = 9$, while the number of people who did not buy chocolate ice cream is $24 - 17 = 7$. Hence the number of people who bought ice cream of both flavours is $24 - (9 + 7) = 8$.

Second solution. If we add the number of those who bought chocolate ice cream to the number of those who bought strawberry ice cream, then those who bought both flavours will be counted twice, hence as the result we will obtain 24 plus the number of people who bought both flavours. Thus the required number is $(15 + 17) - 24 = 8$.

9.64. A total of $100 - 5 = 95$ tourists speak either English or German (or both). Among them $95 - 84 = 11$ don't speak English and $95 - 56 = 39$ don't speak German, so $95 - (11 + 39) = 45$ speak English and German.

9.65. Vitya watered 1010 bushes, of them 1007 by himself, and 3, together with Anya. In the same way, Anya watered 1010 bushes, of them

1007 by herself, and 3, together with Vitya. Together they watered $1007 + 1007 + 3 = 2017$ bushes. The number of bushes that remained unwatered is therefore equal to $2020 - 2017 = 3$.

9.66. The number of those who participate in the drama studio and in the choir, but are not involved in sports, is $10 - 3 = 7$. The number of those doing sports and singing in the choir, but not participating in the drama studio, is $6 - 3 = 3$. The number of those who do sports, participate in the drama studio, but don't sing in the choir, is $8 - 3 = 5$. In order to obtain the number of those who only participate in the drama studio one has to subtract from 27 first the number 3 (those who do everything), then 7 (drama studio and choir) and 5 (drama studio and sports). As a result, we obtain $27 - (3 + 7 + 5) = 12$. Similarly, the number of those only singing in the choir is $32 - (3 + 7 + 3) = 19$, while the number of those only doing sports is $22 - (3 + 5 + 3) = 11$. So the total number of those who participate in something is

$$3 + (7 + 3 + 5) + (12 + 19 + 11) = 60,$$

Therefore, the number of those who don't participate in anything is $70 - 60 = 10$.

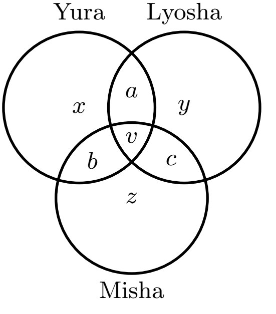
9.67. Let us give each pupil a sticker for attending each session. In all, during the four sessions $20 \cdot 4 = 80$ stickers will be given. Each pupil who attended 3 times will be given 3 stickers, hence the nine pupils who attended 3 times will receive $9 \cdot 3 = 27$ stickers. Similarly, those who attended twice will receive $5 \cdot 2 = 10$ stickers. The 3 who attended only one session each will receive 3 stickers. The remaining $80-(27+10+3) = 40$ stickers are for those who attended all the four sessions, and each of them will receive 4 stickers. Therefore, $40 : 4 = 10$ pupils attended all the sessions.

9.68. In each room, let us gather all flowers of the same type on a separate table (i.e., a table for roses, a table for carnations, etc.). There will be at most 60 such tables (no more than the number of bouquets). Now let us mark one table in each room. In a room with flowers of only one type, there are no unmarked tables, in a room with flowers of exactly two types, there is only one such table, in a room with flowers of all the three types, there are two such tables. The sum $2+3+4 = 9$ is equal to sum of the number of unmarked tables and the number of rooms with flowers of all three types. Indeed, in each such room there are two tables without marks, and when we sum, we count such a room three times. But the number of such rooms is no more than the number of rooms with chrysanthemums and carnations, i.e., than two. So there are at least 7 unmarked tables, hence the number of marked tables (which is equal to the total number of rooms) is at most $60 - 7 = 53$.

9.69. The number of horses is equal to the number of two-humped camels, hence we can pair up the horses and the two-humped camels. In each such pair the number of humps equals the number of animals. The number of humps of the remaining camels is also equal to the number of animals. Therefore the number of humps is equal to the total number of animals in the herd.

9.70. A total of $12 - 9 = 3$ children came with a bucket and a spade along with the bucket. A total of $12 - 2 = 10$ children came with a spade. So $10 - 3 = 7$ children brought a spade, but not a bucket. The required number is $7 - 3 = 4$.

9.71. Any pupil who got into the list I, II, or IV solved at least one problem, so he or she got into the list III, too. If nobody solved any problem, then all the lists are identical: they are all empty. Now suppose that at least one pupil solved at least one problem. If no one solved the first problem, then the lists I and IV are identical (they are empty) and the lists II and III are also identical (they consist of all those pupils who solved the second problem). If nobody solved the second problem, then it is the lists I, II, and III that are the same. If there is a pupil who solved the first problem only, or the second problem only, or both problems, then all the lists differ. If each pupil solved both problems, then the lists I, III, and IV coincide.



9.72. Let us introduce the notation using a Venn diagram (see the figure). For instance, x is the number of stamps that only Yura has, a is the number of stamps that Yura and Lyosha have but Misha has not, etc. In particular, v is the number of stamps that all the three boys have. We know that $x + b < a + v$, $a + y < c + v$ and $z + c < b + v$. Adding these three inequalities, we obtain

$$(a + b + c) + (x + y + z) < (a + b + c) + 3v,$$

hence $x + y + z < 3v$. Therefore, the number v is positive.

9.73. Suppose that the fourth pencil case contains a violet pen. Then it cannot contain either a green pencil (otherwise the first and fourth pencil cases would contain two pairs of identical objects) or a yellow eraser (then the third and fourth pencil case would contain two pairs of identical objects). Thus in the second and the fourth pencil cases there are no identical objects, which does not meet the requirements. Similarly one can prove that the fourth pencil case cannot contain a green pencil or a yellow eraser. Therefore, the fourth case must contain a blue pen (else there will be no coincidences with the third pencil case), an orange pencil (else there will be no coincidences with the third pencil case), and a red eraser (else there will be no coincidences with the first pencil case).

9.74. If we take a 12-car train and disconnect its last car, then we obtain an 11-car train, and if we take an 11-car train and attach to it the remaining car, we get a twelve-car train. Let us check that both these operations transform different trains into different trains. For attaching an extra car to an 11-car train this is obvious. For 12-car trains there are two cases: the last cars are the same, or different. If the last cars are the same, then among the first 11 cars of different 12-car trains there must be a difference. If the last cars differ, then the sets of the first 11 cars are also different, even if one does not take into account their order. Thus we have established a one-to-one correspondence between the set of 12-car trains and the set of 11-car trains.

9.75. If one of the circles comprises the whole class, then this circle possesses the required property. From now on we will assume that there is no such circle. Denote by A a circle with the highest attendance. According to our assumption, there is a pupil X who does not attend the circle A. He and a pupil attending A both attend another circle that we denote by B. All the participants of the circle A cannot also participate in B, else there would be more participants in B than in A. Hence there is a participant in the circle A who, along with X, takes part in a third circle C. The pupil X takes part only in the circles B and C, hence each participant in the circle A also also participates in B or C (else this pupil would not have a common circle with X). The argument concerning X applies to any pupil that is not in A, and so anyone who is not in A participates in both circles B and C. Thus there are exactly three circles and each pupil takes part in two of them. Suppose there are n pupils in the class, then there are $2n$ participants in three circles. So the circle with the highest attendance has no less than $\frac{2n}{3}$ participants.

9.76. The shoes worn by Bom and Bam were not red, hence Bim was wearing red shoes, hence his shirt was red. So Bam's shirt was neither red nor green, hence it was blue. Thus Bom was in a green shirt and blue shoes.

9.77. For instance, such a choice is impossible if all the dresses of two styles are blue, while the dresses of the third style are red and green.

9.78. Consider two pitchers of different shape. If their colours are also different, we are done. If their colour is the same, take a third pitcher with a colour different from that of these two. Its shape differs from the shape of at least one of those two. Taking the third pitcher and the one of the first two that whose shape differs from that of the third, we obtain the required pair.

9.79. Here is an example with Russian names: Andrey Vassilievich Ivanov, Andrey Gennadievich Petrov, Boris Gennadievich Ivanov, Boris Vassilievich Petrov.

9.80. Denote by x the number of participants in the conference who are both mathematicians and philosophers. Then the number of mathematicians is $7x$ and the number of philosophers is $9x$.

9.81. Suppose that in the number $ABCDE$ the first two digits are both not equal to 5. If $B \neq 0$, then to that number we assign the number $BCDEA$, while if $B = 0$, to that number we assign the number $5CDEA$. In this way way we obtain all the five digit numbers not divisible by 5.

Chapter 10. How to Act

10.1. If we take only two balls, it may happen that both are black or both are white. If we take three balls, then, since there are three balls but only two colours, among them there will be two balls of the same colour.

10.2. If we take 13 balls, it may happen that there are 12 white ones and 1 black one. If we take 14 balls, there are at most 12 white ones among them, hence at lest two black ones.

10.3. If we take only 3 jewels, it may happen that we have taken one ruby, one diamond, and one sapphire. If we take 4 jewels, then, since there are 4 jewels but only 3 types of jewels, at least two jewels will be of the same type.

10.4. If we take 30 jewels, then they may happen to be 25 rubies, 4 sapphires, and 1 diamond. If we take 31 jewels, then, since there can be no more than 25 rubies and 4 sapphires among them, at least two of them will be diamonds.

10.5. There are at most 9 black balls in the vase (otherwise one could take 10 black balls). There are at most 11 white balls (otherwise one could take 12 white balls). Since $9 + 11 = 20$, which equals the total number of balls in the vase, one concludes that there are precisely 9 black balls and 11 white ones.

10.6. If among any 3 coins there must be a one-ruble coin, there cannot be more than 2 coins other than one-ruble ones. If among any 4 coins there must be a two-ruble coin, then there cannot be more than 3 coins other than two-ruble ones. Hence among any 5 coins there are at least 3 one-ruble ones (there cannot be more than two other ones) and 2 two-ruble ones (there cannot be more than three other ones). But $2+3=5$, so all the coins have been indicated: 3 one-ruble coins and 2 two-ruble ones.

10.7. The second condition implies that not only the first tank contains at least 26 litres of water, but that the second tank contains at lest 26 litres of water, too. Hence in the first two tanks there is at least 52 litres of water, which is more than the total volume of 50 litres.

10.8. First, using the 3-litre vessel pour 6 litres into the 7-litre vessel. Then fill the 3-litre vessel, use the water in it to fill the 7-litre vessel to the brim, and pour the remaining 2 litres into the pot. Then add 3 litres to the pot using the 3-litre vessel.

10.9. Let the first of two numbers in brackets denote the amount of water in the five-litre vessel, while the second, the amount in the three-litre one. We can apply the following scheme to solve the problem:

$$(5, 0) \to (2, 3) \to (2, 0) \to (0, 2) \to (5, 2) \to (4, 3).$$

10.10. Let the first of two numbers in brackets denote the amount of water in the nine-litre vessel, while the second, the amount in the four-litre one. We can apply the following scheme to solve the problem:

$$(9, 0) \to (5, 4) \to (5, 0) \to (1, 4) \to (1, 0) \to (0, 1) \to (9, 1) \to (6, 4).$$

10.11. Let the first of two numbers in brackets denote the amount of water in the eight-litre vessel, while the second, the amount in the five-litre one. We can apply the following scheme to solve the problem:

$$(0, 5) \to (5, 0) \to (5, 5) \to (8, 2) \to (0, 2) \to (2, 0) \to (2, 5) \to (7, 0).$$

10.12. Let the first of two numbers in brackets denote the amount of water in the seven-litre vessel, while the second, the amount in the five-litre one. We can apply the following scheme to solve the problem:

$$(7,0) \to (2,5) \to (2,0) \to (0,2) \to (7,2) \to (4,5) \to (4,0) \to (0,4) \to (7,4) \to (6,5).$$

There is another possible scheme:

$$(0,5) \to (5,0) \to (5,5) \to (7,3) \to (0,3) \to (3,0) \to (3,5) \to (7,1) \to (0,1) \to (1,0) \to (1,5) \to (6,0).$$

10.13. Let the first of two numbers in brackets denote the amount of water in the 17-litre vessel, while the second, the amount in the five-litre one. We can apply the following scheme to solve the problem:

$$(0,5) \to (5,0) \to (5,5) \to (10,0) \to (10,5) \to (15,0) \to (15,5) \to (17,3) \to (0,3) \to (3,0) \to (3,5) \to (8,0) \to (8,5) \to (13,0).$$

10.14. Let the first of two numbers in brackets denote the amount of water in the sixteen-litre vessel, while the second, the amount in the fifteen-litre one. We can either add 1 l or decrease the amount of water by 1 l:

$$(16,0) \to (1,15) \to (1,0) \to (0,1) \to (16,1) \to (2,15) \to (2,0) \to (0,2) \to (16,2) \to (3,15) \to \dots$$

or

$$(0,15) \to (15,0) \to (15,15) \to (16,14) \to (0,14) \to (14,0) \to (14,15) \to (16,13) \to (0,13) \to (13,0) \to (13,15) \to (16,12) \to \dots$$

10.15. a) First we fill the eight-litre can, then use it to fill the five-litre one. In the eight-litre can 3 l remain, and so if we pour 5 l in the twelve-litre pitcher (which contains 4 litres), it will contain 9 litres. This sequence of pourings can schematically be written as :

$$(12,0,0) \to (4,8,0) \to (4,3,5) \to (9,3,0).$$

b) Using the same notation,

$$(12,0,0) \to (4,8,0) \to (0,8,4) \to (8,0,4) \to (8,4,0) \to (3,4,5) \to (3,8,1) \to (11,0,1) \to (11,1,0) \to (6,1,5) \to (6,6,0).$$

10.16. Let the first of the two numbers in brackets denote the amount of milk in the can, the second one, in the five-litre jug, the third, in the three-litre jug. We can apply the following scheme to solve the problem:

$$(8,0,0) \to (3,5,0) \to (3,2,3) \to (6,2,0) \to (6,0,2) \to (1,5,2) \to (1,4,3) \to (4,4,0).$$

10.17. Let the first of the two numbers in brackets denote the amount of milk in the can, the second one, in the seven-litre jug, the third, in the three-litre jug. We can apply the following scheme to solve the problem:

$$(10,0,0) \to (7,0,3) \to (7,3,0) \to (4,3,3) \to (4,6,0) \to (1,6,3) \to (1,7,2) \to (8,0,2) \to (8,2,0) \to (5,2,3) \to (5,5,0).$$

10.18. Let the four numbers in brackets denote the amount of water in the barrels with the appropriate numbers. We can apply the following scheme to solve the problem:

$$(24,0,0,0) \to (19,0,0,5) \to (8,0,11,5) \to (8,11,0,5) \to (0,11,8,5) \to (0,13,8,3) \to (8,13,0,3) \to (8,13,3,0) \to (8,8,3,5) \to (8,8,8,0).$$

10.19. Let the first number in brackets denote the number of pails of petrol in the big barrel, the second one, the amount of petrol in the nine-pail barrel, the third, in the five-pail barrel. We can apply the following scheme to solve the problem:

$$(a,0,0) \to (a-9,9,0) \to (a-9,4,5) \to (a-4,4,0) \to (a-4,0,4) \to (a-13,9,4) \to (a-13,8,5) \to (a-8,8,0).$$

10.20. Let the first number in brackets denote the number of pails of petrol in the big barrel, the second one, the amount of petrol in the nine-pail barrel, the third, in the five-pail barrel. We can apply the following scheme to solve the problem:

$$(a,0,0) \to (a-5,0,5) \to (a-5,5,0) \to (a-10,5,5) \to (a-10,9,1) \to (a-1,0,1) \to (a-1,1,0) \to (a-6,1,5) \to (a-6,6,0).$$

10.21. Let us use the previous problem's notation in which the first two numbers are the amount of milk in the cans, the third, the amount of milk in the five-litre pot, the fourth, in the four-litre pot. We can apply the following scheme to solve the problem:

$$(10,10,0,0) \to (10,5,5,0) \to (10,5,1,4) \to (10,9,1,0) \to (10,9,0,1) \to (10,4,5,1) \to (10,4,2,4) \to (10,8,2,0) \to (6,8,2,4) \to (6,10,2,2).$$

10.22. The notation (a,b,c) means that there is a litres in the in the barrel, b litres in the scoop, and c litres in a pail. First we take care of the first pail:

$$(18,0,0) \to (14,4,0) \to (14,0,4) \to (10,4,4) \to (10,1,7) \to (17,1,0) \to (17,0,1) \to (13,4,1) \to (13,0,5) \to (9,4,5) \to (9,2,7) \to (16,2,0) \to (16,0,2) \to (12,4,2) \to (12,0,6).$$

Using a similar scheme, we can pour 6 litres in the second pail.

10.23. *First solution.* Let Tania pour the water from the full smaller jug into the bigger one, then fill the smaller one again and fill the big one to the brim. Further let her empty the big jug and pour the water remaining in the small jug into the big one. If the small jug was a three-litre jug, then the big one will now contain 1 litre, otherwise, 3 litres. After that Tania should try to pour all the water from the filled small jug into the bigger one. If it it doesn't overrun, then the small jug is the three-litre one, if the water overflows, then it is a four litre jug.

Second solution. If Tania's bigger jug contained 10 litres, it would suffice to try to pour water in it from the smaller jug three times. If the water overruns, this would mean that the smaller jug is a four-litre one, if it doesn't, then it's three-litre. A similar verification works with a five-litre bigger jug provided Tania empties the five-liter jug after it is filled the first time.

10.24. Let's "start" the two hourglasses simultaneously. When the seven-minute hourglass runs its course, we turn it around and let it run its course for 4 minutes, i.e., until the the eleven-minute hourglass runs out its course. If we then turn around the seven-minute hourglass, it will "run" for 4 more minutes, so that the hourglasses will have been running for 15 minutes, as required.

10.25. Let us weigh 12 kg of sand and put them aside. From the remaining 12 kg we weigh 6 kg and put them aside somewhere else. From the remaining 6 kg we weigh 3 kg and add them to the 6 kg put aside before, obtaining the required 9 kg.

10.26. Let us simultaneously light the two fuses, the first one from one end, and the second one from the second end as well. The second fuse will burn out completely in 30 minutes. When it does, let us light the first fuse from the second end as well. It will burn out in 15 minutes.

10.27. On one plate of the scales place 3 apples and 4 pears, on the other, 5 apples and 3 pears. If we take off 3 apples and 3 pears from each of the plates, the scales will still be in balance, and so one pear weighs as much as two apples. Therefore, a pear weighs more than an apple.

10.28. Four apples weigh more than 4 pears, so one apple weighs more than one pear. Hence 3 apples plus 1 apple weigh more than 4 pears plus one pear.

10.29. In the first weighing, we can measure out 100 grams of rice. If we put the weight on the same plate as the rice, we can obtain 200 gams of rice. Moving these 200 gams to the other plate, we can measure out 400 g. In three weighings, we have obtained 700 grams.

10.30. a) First let us divide the rice in two equal 4.500 kg parts (first weighing). Then divide one of these parts into two equal parts of 2.250 kg (second weighing). In the third weighings, we separate 250 g from one of these parts by using the weights. The required two kilograms remain.

b) In the first weighing, we divide the rice in the two parts such that one part should be in balance with the other plus the 200 g weight. So, we will have 4.400 g of rice on one plate and 4.600 g on the other. In the second weighing, we divide the 4.400 g of rice into equal parts of 2.2 kg. In the third weighing, we remove the extra 200 g using the weight for the second time.

10.31. See the answers section.

10.32. First weighing: on one plate of the scales, we put the ornaments with the engravings 22 g, 23 g, and 24 g, and on the other, 34 g and 36 g. The second plate will be heavier only if the the groups of ornaments have been chosen correctly. And in that case we will know that the remaining ornament weighs 32 g.

Second weighing: On the first plate we place the ornament with engraving 24 g and the one of weight 32 g, on the second plate, we place the ornaments with engravings 22 g and 34 g. In that case the mass on the first plate will be 56 g (if all is correct) or less (if it isn't), and on the second plate, either 56 g or more. Only if the masses of the mentioned four ornaments are indicated correctly, will the scales be in equilibrium. The engravings on the two remaining ornaments are 23 g and 36 g. These two ornaments come from different groups, so if all the other ornaments were not confused, then neither were these two.

10.33. For convenience, let us suppose that the three apples that Julia put on one of the plates are green, while those on the second plate are red. The weight of the three green apples equals half of the total weight of all the apples, hence Bill did not put all the green apples on one plate. Therefore, two green apples will be on the same plate of the scales, while the third will be on the other plate. The weight of two green apples is less than half of the total weight of all the apples, hence two green apples were on the same plate with two red apples. Thus the weight of three red apples, which is one half of the total weight of all apples, equals the weight of two red apples and two green apples that lie on one of Bill's plates. Thus, the weight of the green apple that lies on the other Bill's plate equals the weight of the two red apples that lie on the same plate, with two green apples.

10.34. Let us put one coin on each of the plates. If the scales are in equilibrium, then the remaining coin is counterfeit. If one of the coins is lighter, then it is the counterfeit one.

10.35. Let us put 3 coins on each plate. If one of the groups of three coins is lighter, then the counterfeit coin is in that group. If the scales are in equilibrium, then then the counterfeit coin is one of the remaining three coins. Now one can, using the method of Problem 10.34, in one more weighing identify the counterfeit coin in a group of three.

10.36. Putting two coins on each plate, one can find out which of the pairs contains the counterfeit coin. Putting one of these two coins on each plate, one can find out which one of them is counterfeit.

10.37. Put 3 coins on each plate. If the scales are not in equilibrium, one can act as in Problem 10.35, if it is in equilibrium, it suffices to compare the weights of two remaining coins.

10.38. Let us put one coin on each of the plates. If the scales are in equilibrium, then both these coins are true. If not, then both remaining coins are true. Let us put one of the true coins on one plate, and one of the coins of the suspect pair, on the other. If the scales are in equilibrium, then the counterfeit coin is not on the scales. If the scales are not in equilibrium, then the counterfeit coin is on the scales.

10.39. Let us mark the coins by the letters A, B, and C. First compare the coins A and B. If their weight is the same, then the counterfeit coin is C. Comparing it with one of the two other coins, we can find out if it is lighter or heavier than the true coins. If the weights of A and B differ, then the remaining coin C is true. Let us compare the lighter one of the coins A and B with C. If their weights are the same, then the heavier of A and B is counterfeit and heavier than the true coins. If their weights are different, then the lighter of A and B is counterfeit and it is lighter than the true coins.

10.40. Let us put a 5-kopeck coin on the the left plate of the scales, and a 2-kopeck and a 3-kopeck coin, on the right one. If the scales are in equilibrium, then the 1-kopeck coin is counterfeit. If the scales are not in equilibrium, put the 3-kopeck coin on the left plate and the 1-kopeck coin and the 2-kopeck coin on the right plate. If now the scales are in equilibrium, then the 5-kopeck coin is counterfeit. If in the second weighing the same plate as in the first weighting is the lighter one, then the 2-kopeck coin is counterfeit (it remained in the same plate), in the opposite case, it is the 3-kopeck coin (it has moved to the other plate).

10.41. Let us divide the coins into three groups: two of them consist of 4 gold coins and 5 silver ones, the third, of 5 gold coins and 4 silver ones. In the first weighing, we compare the first two groups. If their weights are equal, then the counterfeit coin is in the third group. If not, then the counterfeit coin is either among the 4 gold coins on the lighter plate or among the 5 silver ones on the heavier plate. In the second case (when the weights of the two first groups are not equal), let us unite the 4 suspicious gold coins and 5 suspicious silver coins. We then divide these $5 + 4 = 9$ coins into three groups: two groups of 1 gold coin and 2 silver coins, and one group of 2 gold coins and 1 silver coin. Further we detect the group containing the counterfeit coin as we did in the first weighing. If the counterfeit coin is part of the group with 1 gold and 2 silver coins, we perform the third weighing, comparing two silver coins. Then the heavier one is counterfeit, or (if their weights are equal), the gold coin is counterfeit. The counterfeit coin in the group of 2 gold and

1 silver coins is determined similarly. The first case (the counterfeit coin is among the group of 5 gold and 4 silver coins) is considered similarly.

10.42. The hikers can act as follows:

1) the 45 kg and 50 kg people get on the boat and cross the river;
2) the 45 kg person brings the boat back;
3) the 80 kg hiker gets on the boat and crosses the river;
4) the 50 kg person gets on the boat and brings it back;
5) the 45 kg and 50 kg people get on the boat and cross the river.

10.43. First the boys ride the boat to the opposite shore. One of them stays there, the other one rides back. Then one soldier crosses and gets out, the boy returns, then once again the boys ride the boat to the opposite shore, and so on.

10.44. First the goat must be taken across. After that the boat must be returned, and there are two variants of further actions. Take the cabbage (resp. wolf) across, return with the goat, leave the goat on shore, take the wolf (resp. cabbage) across and then take the goat across.

10.45. Let us indicate in brackets who is crossing in the boat. The crossing can be performed as follows:

$$\text{PPGG(PG)},\ \text{PPGG(P)G},\ \text{PGG(PP)G},\ \text{PGG(P)PG},\ \text{GG(PP)PG},\\ \text{GG(G)PPP},\ \text{G(GG)PPP},\ \text{G(P)PPGG},\ \text{(PG)PPGG}.$$

10.46. First the father and mother cross the bridge together, then the father returns with the flashlight, then the grandmother crosses the bridge with the grandson, then the mother returns with the flashlight, and finally the father and the mother cross the bridge together. All this will take up $2 + 1 + 10 + 2 + 2 = 17$ minutes.

10.47. Denote the knights by A, B, and C and their squires by a, b, and c. First the squires a and b cross the river. Then b returns and takes c across; thus abc have crossed. Then c returns, and A and B cross; thus $AaBb$ have crossed. Further a returns, C and c cross; thus $ABbCc$ have crossed. Finally, b returns and crosses with a.

10.48. Let us denote he missionaries by M, m, m, the cannibals by K, k, k (the capital letters indicate the missionary and the cannibal who know how to row). First K takes k, and then another k, across; after that, K returns (thus $KMmm$ are now on the initial side of the river, and kk have crossed). Then Mm cross and Mk return ($KkMm$, km). Next Mm cross, K returns (Kkk, Mmm). Now K takes k across, returns, and takes the remaining cannibal across.

10.49. Ivan can divide the prisoners into 20 pairs and one triplet and ask Koschey to tell each that all the prisoners except those paired with him (or those in the same triplet) are werewolves. Then the prisoners can get back to shore as follows. Let us call the prisoners of the triplet A, B, and C. First A and B cross to the shore and A returns. Next one of the pairs crosses and B returns. As the result, one pair of prisoners has reached the shore, while all the others and the boat are on the island. Similarly, all the remaining pairs cross. Then A and B cross, A returns and takes C across. After that C can return to pick up Ivan.
10.50. For example, the fox can three times eat off 1 g from each of the 5 g and 11 g pieces. After that, one 2 g piece and two 8 g pieces will remain. Now the fox can six times eat off 1 g from each of the 8 g pieces.
10.51. Let us turn over the first three coins. Then the first two coins will be heads up, and the last three, tails up. Now we turn over the last three coins, and all five coins will be heads up.
10.52. Let us take the first key and successively try to open the suitcases. If one of the first five suitcases opens, we put it aside with the key; if not, the key must fit the sixth suitcase, and we put it aside with the key. Now we have spent 5 attempts or less and found the key fitting one of the suitcases; we are left with 5 suitcases and 5 keys. Now in at most 4 attempts we will find another pair (key, suitcase), and only 4 suitcases and 4 keys will be left. Continuing this procedure, we will find 5 fitting pairs (key, suitcase) in at most $5 + 4 + 3 + 2 + 1 = 15$ attempts. The remaining key and suitcase will fit each other automatically.
10.53. No less than $(15 \cdot 4 \cdot 5) : 12 = 25$ minutes are needed, but for that none of the smiths must ever stand idle. Let us arrange the horses in three circles with 5 horses in each. Four smiths come up to the first 4 horses in one of the circles and shoe (in five minutes) one foot of each one. Then, moving clockwise, each of the smiths comes up to the next horse and so on. When the smith who shod the first horse will have shod the fifth horse, all the horses in the circle will be shod. The horses are standing in three circles and there are also three quadruples of smiths.
10.54. a) With each of the first three blows Ivan should cut off cut off 1 tail; after that, 3 heads and 6 tails will remain. With each of the next three blows he should cut off 2 tails; after that, no tails and 6 heads will remain. Finally, with each of the last three blows he should cut off 2 heads, and after that nothing will remain.

b) With the last blow, Ivan must cut off 2 heads (that's the only blow after which nothing grows back). Hence he must act so as to add an odd number of heads (the dragon already has 3 heads, but that number must become even). A head can be obtained by cutting off two tails. So Ivan must act so that the total number of tails should be divisible by 2, but not by 4. The dragon already has 3 tails, the only possibility to obtain a tail is to cut one off since two will grow back. Therefore, Ivan must cut off 1 tail 3 times. Now there are 6 tails. These tails can be cut off by 3 blows. No tails will remain, but 3 new heads will appear. Thus there will be 6 heads. And Ivan can cut them all off in 3 blows.

10.55. a) Let us pick 2 bananas 7 times. One banana and 27 oranges will remain. After that let us pick a banana and an orange 27 times, and only 1 banana will remain.

b) No matter how one picks the fruit, the number of bananas will remain odd (indeed, if one picks one piece of fruit or two pieces of different fruits, the number of bananas remains unchanged, and if one picks two pieces of the same fruit, it either remains unchanged or decreases by two). Since the initial number of bananas is odd, the only fruit that remains must be a banana.

c) It follows from the previous argument that no matter how we pick the fruit, the number of bananas cannot become zero.

10.56. a) +9 (11th floor), −7 (4th floor), +9 (13th floor), −7 (6th floor), +9 (15th floor), −7 (8th floor), −7 (1st floor);

b) +9 (10th floor), −7 (3rd floor), +9 (12th floor), −7 (5th floor), +9 (14th floor), −7 (7th floor), +9 (16th floor), −7 (9th floor), −7 (2nd floor).

10.57. a) Suppose the fox divided the sweets into three piles containing 10, 10, and 80 sweets. If she takes the 80-sweets pile, she will get 80 sweets and the cubs will not complain. If she gets the 10-sweets pile, the cubs will ask her to equalize their shares, and she will eat 70 extra sweets.

b) After the sharing, the cubs will get an equal number of sweets, so the total number of sweets eaten by them is even. The number 100 is even as well, hence the fox in any case will get an even number of sweets.

10.58. First one should place the entire second stack on top of the first. Then one takes off a part of the books so that Volume 1 should be at the bottom and forms a second stack with them. After these two operations, Volume 1 will be at the bottom of the second stack. Next, one moves all the books except Volume 1 back to the first stack, from which one removes the part that begins with Volume 2, and places this part on top of Volume 1. Now, after these two operations, the two lower books in the second stack will be in correct order. Then, using two operations each time, one places Volume 3 on Volume 2, Volume 4 on Volume 3, and so on. After 18 such operations Volumes 1 to 9 will be in correct order in the second stack. If Volume 10 is also there, we are done, otherwise one places it on the second stack, so that in 19 operations one has arranged the books as required.

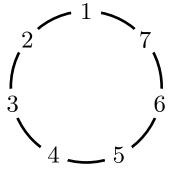
10.59. In all we are to fry 6 sides of the pieces of bread. Simultaneously, we can only fry two sides, so at least 3 minutes will be needed. And this can actually be done in 3 minutes as follows. First we place two pieces on the frying pan, a minute later we turn the first one over and take off the second one, replacing it by the third. One more minute later, we take off the fully fried first piece, replace it by the second piece with the non-fried side down, and turn over the third piece which is on the pan. A minute later all three pieces will have been fried on both sides.

10.60. The spy can act as follows. He begins going around the circle from some room (call it the first) and turns on the light there (if it wasn't on already). He continues along the circle, counting the rooms and turning on the light in each new room. Suppose the spy enters a room in which the light is on, say, the 10th one. Then he turns off the light in it and returns to the first room (moving back by 9 rooms). If the light is off in the first room, then the 10th room was actually the first, and the number of rooms in the circle is 9. If the light is on, the spy returns to the 10th room and continues his circling as before.

10.61. For the first 9 numbers, we write: 11, 22, 33, 44, 55, 66, 77, 88, 99. If Petya writes a plus sign next to one of these numbers, it has been guessed. If there are no pluses, there must be at least one minus, because we have listed all the possible versions of first digits. If there is a minus sign next to exactly one number, then its first digit coincides with the first digit of the number chosen by Petya, and its second digit is 0. If there are two minuses next to, say, the numbers 22 and 55, then the chosen number is either 25 or 52. Adding one of those two numbers to the previous list, one will learn from Petya's answer which of them was chosen by him.

10.62. Let us number the doors 1, 2, 3, 4, 5, and 6, say, clockwise. In his first attempt, Prince Ivan checks doors 1, 2, and 3. Then if he does not succeed in leaving the room, that means the doors were locked. Whatever Baba-Yaga locks and unlocks after that, door number 2 will remain locked. In his second attempt Ivan checks the doors 3, 4, and 5. If he still does not succeed in leaving the room after that, this means that door 5 was locked and, having in mind that door 2 is locked, too, we can assert that doors 3 and 4 will remain locked whatever Baba-Yaga will do. Now Ivan checks the doors 5, 6, and 1. Arguing in a similar way, we see that if Ivan has not left the room, then doors 4, 5, and 6 will remain locked, and so one of the doors 1, 2, and 3 will be open; since Ivan has the right to check them, he will leave the room.

10.63. Each gnome will see all the caps except two: the hidden one and his own, so each gnome must name one of these two colors. Let us number the colours from 1 to 7 and arrange these numbers in a circle (see the figure).



First strategy. Of the two colours the gnome does not see, each gnome names the one for which the distance form it to the other one, if moving along the circle clockwise, is shorter (for example, if a gnome does not see the colours 2 and 5, he is to name 5 and not 2). Then the three gnomes will guess right and the other three will be mistaken. For instance, if the cap of colour 1 is hidden, then the gnomes with caps 2, 3, and 4 will guess correctly.

Second strategy. If a gnome does not see two colours of the same parity, then he names the colour with the greatest number; if the colours he does not see are of opposite parity, then he names the one with the least number. Whatever the colour of the hidden cap, exactly three gnomes will name it. For instance, if the hidden cap is of colour 3, then the gnomes with numbers 1, 4, and 6 will name its colour.

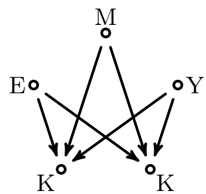
10.64. 1. Dorothy writes the number 1 on one of the coins and asks about two other coins, paying with coin number 1. It the Wizard says that there are no counterfeit coins among them, then all three coins are true. Now if the Wizard says that there is a counterfeit coin among the presented ones, then one of these three coins is counterfeit, so the two remaining coins are true. In any case, of the four coins that she has, Dorothy, at this stage, knows two true ones; let her write the numbers 2 and 3 on them, and on the two other remaining coins, let her write the numbers 4 and 5.

2. Dorothy gives the Wizard the coin with number 2 (which is, as she knows, true) and asks the Wizard about the coins with numbers 3 and 4 (the latter's quality is unknown). The answer will be correct, hence either the counterfeit coin is in this pair (in that case it is the coin

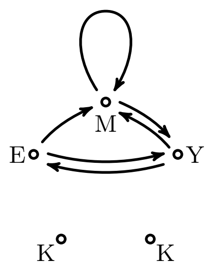
number 4) so Dorothy doesn't need to ask any more questions, or both coins are true (and then the counterfeit coin has the number 1 or 5).

3. Now Dorothy gives the true coin the number 3 to the Wizard and asks about the coins number 4 and 5. The answer is correct, hence if among them there is no counterfeit coin, then the counterfeit coin has the number 1, while if there is a counterfeit coin among them, then the counterfeit coin has the number 5.

10.65. a) Ivan should instruct the princesses to answer "Yes" to the questions about Koschey's daughters (as shown in the figure). Then Koschey's daughters will have been named no less than three times, and the princesses no more than twice, so Prince Ivan will recognize them.



b) Suppose the eldest princess names the middle princess and the youngest one, the youngest one names the middle one and the eldest one, the middle one name herself and the youngest princess (see the figure). Then Ivan will immediately recognize the middle princess as the only person who named herself and who was named by two other persons. After that Ivan will recognize the youngest princess (she was named by the middle princess) and the eldest princess (she was named by the youngest princess).



10.66. Assume that we did not replace at least two lightbulbs. If one of them was blown, then we will not be able to determine which one. Hence, in order to certainly determine the burned out lightbulb, we must at least remove three of them (30 seconds) and replace them by three others (30 more seconds). Let us prove that 60 seconds will suffice.

Let us remove the first lightbulb and replace it by the spare (20 seconds). If the garland lights up, then we are lucky and 20 seconds were enough. If it does not, then the only blown lightbulb is still in the garland and one spare is in our hands. Then we remove the second lightbulb and replace it by the former first lightbulb (40 seconds in all). If we are out of luck again, then we unscrew the third lightbulb and replace it by the former second (60 seconds in all). If the garland still does not light up, then the blown lightbulb is the last one.

10.67. After each move, the number of heaps increases by 1, and the game will be over when there are 45 heaps, of one stone each. So, the game will be finished in $45 - 3 = 42$ moves. Thus, it will be the second player who makes the last move and wins.

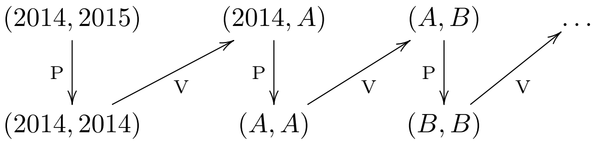
10.68. The first should take 100 sweets from the second pile, and then repeat the moves of the first, keeping the two piles equal.

10.69. Let us call the number of occurrences of a letter in an inscription the multiplicity of the letter. After Betsy's move, the letters A and O have multiplicity 3, the letters C and Q are of multiplicity 2, while each of the letters G, K, L, M, H, P, R, T, and Y has multiplicity 1. In order to win, let Suzy erase any letter of multiplicity 1. Then, for each multiplicity, the number of letters of this multiplicity will become even. Further Suzy is to play so that, after each her move this property should still hold: for each multiplicity, the number of letters of this multiplicity should remain even. To do that, in response to any move by Betsy, Suzy should erase the same number of letters of the same multiplicity as Betsy did. For instance, if Betsy erases 3 letters A, then Suzy is to erase 3 letters O, while if Betsy erases one letter D, then Suzy can erase, say, one letter I. Thus it is Suzy who will make the last move and win. Visually, this strategy can be represented as follows. Let us reorder the letters as follows: "AAADDIIKLMHGPRTYCCQQOOO". Then in her first move Suzy erases the letter G, and further makes moves symmetric to Betsy's with respect to the middle of this "word".

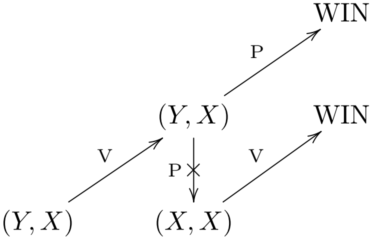
10.70. Freddy can always secure two bills for himself: he knows the place where the very last digit must be written, so he can name that digit so that it differs from the digit at the same place in some other bill. Then these two bills will have different numbers and Freddy can take them both.

Let us prove that Johnny can work things out so that there will be no more than two bills with different numbers. To that end he can place the bills next to each other so that the slots for the digits form a table. When Freddy names the digit 1, Johnny writes it in the leftmost column in which there is still an empty slot, and when Freddy names the digit 2, Johnny writes it into the rightmost column (in which there is an empty slot). If in some column both digits occur, this means that all other columns are filled: ones to the left, twos to the right. Hence, if the numbers differ, they differ only by the digit from that column. Since Freddy uses only two digits, there are no more than two different numbers.

10.71. Let Petya replace 2015 by 2014 in his first move, and in his further moves equalize the two numbers (he can always do that by repeating with the other number whatever Vassya did with the changed number):



If Petya does this during the whole game, then at some moment Vassya will transform one of the numbers into a one-digit one and win. But let us look at this moment more carefully. If Vassya won by replacing one of the pair of numbers (X, X) by a one-digit number, then just before that, when it was Petya's move, the number X was already on the board. So at that moment Petya can replace it by a one-digit one, thereby winning:



Indeed, by his winning move Vassya would either divide the number X by 2 or subtract from X one of the non-zero digits of X; but Petya, who has the numbers X and Y at his disposal, can do the same at the previous move (instead of equalizing the numbers). Petya's strategy can be described as follows. If a number can be replaced by a one-digit number, then do it; otherwise, equalize the two numbers.

10.72. If the pirates had an unbounded amount of money, John would have a simple winning strategy: after any move by Bill, John should complete the number of coins placed on the table by Bill to 4, then the number of coins on the table after each of John's moves will be divisible by 4, so John will place the one hundredth coin on the table at his 25th move. But the pirates have only 74 coins each, so if Bill keeps placing 1 coin at each move, then John will have to place 3 coins at each move, and so he will run out of coins for his 25th move. Therefore, in order for his strategy to remain winning, he has to place 1 or 2 coins at certain moves. Let us show how he can do it.

If at his first move Bill places 3 or 2 coins, then John places one or two coins respectively and then keeps the number of coins on the table divisible by 4; it is easy to see that in this case he will not run out of coins at the 25th move, so the original strategy works. Suppose Bill places 1 coin at his first move. Then John's return move should be to place 1 coin. Further three cases for Bill's second move are possible.

1. Bill places 1 coin. Then John also places 1 coin. So there will be 4 coins on the table, and then John will be able to implement the strategy described above.

2. Bill places 3 coins. Then John also places 3 coins. Then there will be 8 coins on the table, and so John has already placed less than 3 coins; thus he will be able to implement the strategy described above.

3. Bill places 2 coins. Then there will be 4 coins on the table. Then John should place 1 coin. Bill must not allow the number of coins to be divisible by 4 after John's move, so Bill will have to place 3 coins. At each of the subsequent moves, John will place 1 coin, forcing Bill to place 3 coins. But Bill will not be able to place the one hundredth coin, because for that he would need to have 75 coins initially. Therefore the one hundredth coin will be placed by John.

10.73. Prince Ivan can act so that each of Koschey's moves will be uniquely determined: all the numbers will have appeared before, or will be too large so that Ivan at that moment will not have that many ingots.

a) We will denote the moves as $X(Y)$, where X is the number of ingots moved and Y is the number of ingots in Ivan's sack after the move. Now the game may proceed as follows: 2 (2), 1 (1), 3 (4), 4 (0), 6 (6), 5 (1), 7 (8), 8 (0), 10 (10), 9 (1), 11 (12), 12 (0), 13 (13).

All the possible moves having been made, Ivan has all the ingots and can leave with them.

b) Suppose Ivan acts as in item a). After the move 13 (13), the only move not made is 14, but it is impossible, hence Ivan can leave with 13 ingots.

Let us prove that Ivan can't leave with 14 ingots. Assume that at some moment there were 14 ingots in the sack. This means that there are no ingots in the chest and the previous move was Ivan's. But then an odd number of moves had been done, so one of the numbers 1 to 14 had not appeared, Koschey could have made a move with that number, so Ivan cannot yet leave with the gold.

Chapter 11. Extra Problems

11.1. See the answers section.

11.2. Begin with the four central squares, then fill in the diagonals.

11.3, 11.4, 11.5, 11.6. See the answers section.

11.7. When one plus is replaced by minus (or one minus is replaced by plus), the parity of the value of the expression does not change. Hence this parity does not change after several such replacements either.

11.8. a) In the exchange, the parity of the number of coins does not change, hence if initially the number of coins was odd, then it will be odd at the end.

b) In the exchange, the remainder under division by 4 of the number of coins does not change. Hence if initially the remainder was 1, then it cannot become equal to 3 at the end.

11.9. Initially, the page was torn in two pieces, and then, after each act of tearing, Johnny added 8 pieces and Freddy added 12. Hence the remainder of the total number of pieces under division by 4 does not change. The remainder of number 99 is 3 under division by 4, therefore 99 is not all.

11.10. Each time the signs in 8 cells change, the product of all the numbers in the table does not change. Initially it was negative, so it will never become positive.

11.11. When we change the colours in a column or a row, the parity of the total number of black cells does not change, and initially it was odd.

11.12. At every move the knight changes the colour of the square, i.e., the colours of the squares alternate. Therefore it can return to the initial square only after an even number of moves.

11.13. In order to visit all the squares of the chessboard, the knight has to make 63 moves. Since after each move of the knight the colour of the square changes, after 63 moves it will be on a square of different colour than a1, but the squares a1 and h8 are both black.

11.14. At each non-diagonal move the colour of the square in which the king is located changes, whereas at diagonal moves it remains the same. Since the king visited all the squares of the board and returned to its initial position, the colour of the square changed from white to black as many times as it changed from black to white, and so the king made an even number of non-diagonal moves. The number of diagonal moves is obtained by subtracting the number of non-diagonal moves from 64, so it is also even.

11.15. After each operation the sum of all the numbers decreases by one, and the quantity of numbers decreases by one as well. Hence the difference between the sum of all the numbers on the board and their quantity does not change. Initially this difference is $(1+2+\ldots+20)-20 = 190$. After 19 operations, when only the number p remains on the board, the difference equals $p-1$. Therefore $p = 191$.

11.16. Initially, the sum of the four numbers is odd, whereas the sum of four equal numbers is even. It is impossible to obtain an even sum from an odd one by adding several twos to it.

11.17. *First solution.* Each glass must be turned over an odd number of times, and the number of glasses is odd. Hence, one is to perform an odd number of turnovers. However, at each step an even number of glasses is turned over. Therefore, it is impossible to turn all the glasses right side up.

Second solution. Suppose we turned over 4 glasses of which k were standing upside down and $4-k$, right side up (k can take the values from 0 to 4). After that, of these 4 glasses k will be standing right side up, $4-k$, upside down. Thus, to the number of glasses standing upside down, an even number $(4-k)-k = 2(2-k)$ will be added, so the parity of this number will not change. Hence at any moment there will be an odd number of glasses standing upside down, as initially, when that number was 7.

11.18. Suppose the numbers n and m have a common divisor $d > 1$, and the sum of the numbers written out is not divisible by d. Let us show that in this case all the numbers cannot be made equal. Indeed, when 1 is added to n of these numbers, the sum of all the numbers increases by n. Since n is divisible by d, the sum of numbers will have the same remainder under division by d, hence it will never become divisible by d. If at some moment all the m numbers become equal, then the sum of all the numbers is divisible by m, and therefore by d.

11.19. The number of ones remains even after every move. Initially the number of ones was even, hence this number will always be even. In particular, a single 1 cannot be the last remaining number.

11.20. In each of the initial heaps the number of stones is odd. If each of the heaps contains an odd number of stones, this property will be preserved after each of the two permitted operations. So none of the heaps can become empty.

11.21. Let us number the trees from 1 to 6 in order.

a) *First solution.* Let us assign to each sparrow the number equal to that of the tree on which it is initially sitting. Then the sum of the numbers assigned to the sparrows is an invariant. Initially it is equal to $1 + 2 + \ldots + 6 = 21$. If all the sparrows had gathered on one tree, then the sum of the numbers assigned to the sparrows would be 6 times the number of that tree. But 21 is not divisible by 6, therefore the sparrows can't gather on one tree.

Second solution. Let us check that the number of sparrows on the trees with even numbers will always be odd (as well as on the trees with odd numbers). If two sparrows together cover an even number of metres, then the number of sparrows on the even-numbered trees does not change. If two sparrows cover an odd number of metres, then the number of sparrows on the even numbered trees can:

1) decrease by 2 (if both sparrows were sitting on even-numbered trees);

2) remain the same (if one sparrow was sitting on an even-numbered tree, the other, on an odd-numbered one);

3) increase by 2 (if both sparrows were sitting on odd-numbered trees).

Now if the sparrows had gathered on one tree, there would be an even number of sparrows on even-numbered trees.

b) To gather all the sparrows on the fourth tree, the sparrows from the first and seventh, the second and sixth, the third and fifth trees should simultaneously fly to the fourth tree.

11.22. Assume for convenience that the corridor goes from North to South. Then Vassilissa can place Koschey in the northernmost room and lean the guards as follows, North to South: on the West wall, on the East wall, on the West wall. Let us show that, no matter how Koschey moves, all the guards will never lean against the same wall. Note that at any moment the following condition holds: the guards who are to the South of Koschey remain in their initial positions, while the positions of those to the North of Koschey differ from their initial positions. Indeed, this condition holds initially and it will still hold after Koschey has moved from a room to an adjacent room. Hence, if at some moment Koschey is in the northernmost room, then all the guards are in the position WEW. If he is in the second room, then the first guard (i.e., the guard from the northernmost room) has changed his position, while the two others remain in their initial positions, so that the guards are EEW. Now if Koschey is in the third room, the guards are EWW. Finally, if Koschey is in the southernmost room, then all the guards have changed their positions, i.e., their position is EWE. Thus, the guards will never lean on the same wall.

11.23. Eleven players are to be distributed among seven days of the week. Then to one of these days, more than one player will correspond.

11.24. Assume that on each of the 365 or 366 days of the year less than 10 000 Muscovites were born. Then there is no more than $366 \cdot 9999 <$ 3 660 00 inhabitants in Moscow, which is not the case.

11.25. The age of the hikers is one of the 16 numbers: $20, 21, 22, \ldots, 35$, and there are more than 16 hikers.

11.26. Suppose it is possible. Let us order the piles in the increasing order of the number of marbles. Then in the first pile there must be at least one marble, in the second, at least 2, in the third, at least 3, and so on. In all there should be no less than $1 + 2 + 3 + \ldots + 9 = 45$ marbles. But there are only 44 marbles.

11.27. If there are 13 girls in the class, then the number of boys with whom each of them is friends may be anything from 0 to 12 (13 variants), which meets the condition. Now if there at least 14 girls, then there are no more than 11 boys in the class, so there are at most 12 variants for the number of boys who are friends with them (from 0 to 11). Therefore at least two girls will have the same number of boys with whom they are friends.

11.28. Since there are 11 possible remainders of division a natural number by 11, among any 12 two-digit numbers there are two with equal remainders. Their difference is nonzero and a multiple of 11; since it is less than 100, we are done.

11.29. Among the numbers $2, 2^2, \ldots, 2^{20}$ there are two numbers that have the same remainder under division by 19. Their difference is divisible by 19.

11.30. Since among every 4 socks at least 2 belong to one child, there can be no more than three children. None of the children can have more than 3 socks (otherwise one can find 5 socks among which 3 belong to the same child). In all, the mother found 9 socks, hence there cannot be less than three children. So, the socks belonged to precisely three children and to each of them three of the found socks belong.

11.31. Suppose there are n persons in the group. Then each of them has from 0 to $n-1$ friends. Thus, the number of friends can take n different values: $0, 1, 2, \ldots, n-1$. Hence if all n persons have different numbers of friends, then in the group there will be one person who has $n-1$ friends. On the other hand, if there is a person with $n-1$ friends, then they are friends with everybody, and so there is no one with 0 friends. A contradiction.

11.32. Choose one of the 6 persons. Then among the other 5 they have either (at least) 3 acquaintances or (at least) 3 persons with whom he is unacquainted. In the first case, among these 3 persons there are either 2 acquaintances and then, together with the chosen person, they form the required triple, or all three are pairwise unacquainted. The argument in the second case is similar.

11.33. If there is an equal number of left and right boots in one of the given sizes, they will form 100 pairs meeting the requirement. So further we will assume that in each of the sizes there are either less left than right boots or vice versa. There are two types of boots (left and right) while there are three sizes. Hence one of the types of boot, say the left-foot type, will form the minority in at least two sizes. If these sizes are, say, 41 and 42, then there are no more than 200 left boots of size 43, hence there are at least 100 left boots of sizes 41 and 42. With them, one can compose at least 100 pairs with the required properties.

11.34. If there were no more than 6 coins of each of the four types, then there would be no more than a total of $6 \cdot 4 = 24$ coins, whereas there are 25.

11.35. We know that that there are $10 - 3 = 7$ pupils who solved $35 - 6 = 29$ problems. If each one of them solved less than 5 problems, then together they solved no more than $4 \cdot 7 = 28$ problems. Hence one of them solved at least 5 problems.

11.36. Suppose each of them has a classmate namesake. Then all the Lyoshas are in the same class (else for one of them we would be unable to find namesake in the same class). Similarly, all the Vanyas are in the same class. Since only four pupils from each class belong to this group, the Lyoshas and Vanyas are in different classes. Hence one of the Artyoms is in class 6“A”, while the other, in class 6“B”, a contradiction.

11.37. Glasha and Natasha together ate 9 bowls, hence one of them ate no less than 5 bowls. But then Masha, who ate more than each of them, must have eaten no less than 6 bowls. In all, these three monkeys ate no less than $9 + 6 = 15$ bowls. Hence Dasha got only one bowl.

11.38. Assume that the weights can be divided into 10 piles that meet the requirements. The sum of masses of all the weights is 5050. Hence, the mass of the heaviest pile is at least $5050 : 10 = 505$. In the collection, there are no weights of mass greater than 100, hence in that pile there are at least 6 weights. Therefore, the total number of weights is at least $6 + 7 + 8 + \ldots + 15 = 105 > 100$, a contradiction.

11.39. The first condition implies that the movie theatre has no more than 29 rows. Indeed, if there were at least 30 rows, then, obviously, a class of 30 pupils could have been placed with no more than one in each row. The second condition implies that there are no less than 29 rows. Indeed, if there were at most 28 rows, then no more than two rows will be empty, even if no two pupils sit in the same row. Therefore, the movie theatre has 29 rows, and it is clear that in this case it has the required properties.

11.40. Taking 8 snapshots is unsafe: 3 woodpeckers and 5 wagtails could fly away, and only two birds of these two types would remain. Let us show that 7 snapsholds will do. After 7 shots, $20 - 7 = 13$ birds of at most three types remain in the atelier. Hence there are at least 5 birds of one of the types. On the other hand, there are no more than 8 birds of that type, because initially there were no more than 8 birds of each type. Therefore, together there are at least 5 birds of the two remaining types, and so there are at least 3 birds of one of these two types.

11.41. Let us draw an arrow to each female cat from the tomcat sitting next to the female cat if the tomcat is heavier. Since for each female cat there is an arrow directed at her and there are 19 female cats, there are at least 19 arrows. No more than two arrows can issue from any tomcat, because those arrows are directed only to neighbouring female cats. It there is no arrow issuing from some tomcat, then there are no more than $2 \cdot 9 = 18$ arrows, a contradiction. Hence there is an arrow issuing from any tomcat, as required.

11.42. Since the sum of the two-digit number AA and the one-digit number A is a three-digit number, it follows that A = 9 and BCD = 108. In the product $\mathrm{E} \cdot \mathrm{F} \cdot \mathrm{G} \cdot \mathrm{G} \cdot \mathrm{F} \cdot \mathrm{H} \cdot \mathrm{I} \cdot \mathrm{J}$ six other digits are used. Therefore, the digits 2 and 5 are among them and so the product ends with zero.

11.43. Let us divide all the remainders of the division by 100 into 50 groups: $\{1, 99\}$, $\{2, 98\}$, …, $\{49, 51\}$, $\{0, 50\}$. Since there are more than 50 numbers, we can find two numbers x and y whose remainders will fall into one of the groups. If it is one of the first 49 groups, then either $x+y$ or $x-y$ are divisible by 100. If it is the last group, then both $x-y$ and $x+y$ are divisible by 10. In any case, $x^2 - y^2 = (x-y)(x+y)$ is divisible by 100.

11.44. The natural numbers from 1 to 24 into can be divided into 12 pairs. The magicians can agree in advance how to do this. For instance, $\{1,24\}$, $\{2,23\}$, $\{3,22\}$ etc. Among the 13 cards chosen by the spectator, there are two on which numbers from the same pair are written. It is precisely those that the first magician should return to the spectator. In that case, the spectator will have to add a card not from this pair. The second magician will then recognize that card.

11.45. Adding the numbers, one sees that $1+2+4+8+16+32+64 = 127$. The binary notation of any number which is less than $2^7 = 128$ contains at most 7 digits. To pay any sum from 1 to 127 rubles one can write it in binary notation and the purses corresponding to the ones in that notation should be given out.

11.46. The binary notation of any number which is less than $2^{10} = 1024$, contains no more than 10 symbols (binary digits). Prepending the appropriate number of zeros, we will assume that all the numbers under consideration contain 10 binary digits. About each of those digits, Vassya can ask whether it is equal to 1 or not. Having found all the binary digits of a number, Vassya can learn what that number is.

These questions may be stated so that Misha will understand them even if he is unaware of the binary system.

For simplicity, let us explain this for the case of numbers whose binary notation contains at most 3 symbols. Suppose Misha chose the number $110_2 = 6$. First Vassya asks whether the number is greater than $3 = 011_2$ (this is the greatest three-digit binary number of which the leading digit is zero). Thus he will learn that the first digit is 1. Then he asks whether the number is greater than $5 = 101_2$ (this is the greatest three-digit binary number of which the leading digit is 1 and the second digit is 0) and thereby learns that the second digit is also 1. Finally, he asks whether is it true that the number is greater than $6 = 110_2$ (at this stage, it remains to choose between 110_2 and 111_2). Thus he will learn that the last binary digit is 0.

11.47. The number 61 is greater than $54 = 2\cdot 27$ and less than 81. Hence we can write $61 = 2\cdot 27+7$. It is also clear that 7 is greater than $6 = 2\cdot 3$ and less than 9. Hence $61 = 2\cdot 27 + 2\cdot 3 + 1 = 2\cdot 27 + 0\cdot 9 + 2\cdot 3 + 1$.

11.48. Let us use the representation

$$61 = 2021_3 = 2\cdot 27 + 2\cdot 3 + 1,$$

obtained in Problem 11.47. To the number $61 = 2021_3$, let us add the number

$$1 + 3 + 9 + 27 + 81 = 11111_3.$$

As a result, we obtain 20202_3. Thus,

$$61 = 20202_3 - 11111_3 = 81 - 27 + 9 - 3 + 1.$$

11.49. By using the four weights indicated in the answer section, one can balance any integer number of kilograms from 1 to $1111_3 = 40$ inclusive by using the same method as in the solution of Problem 11.48. If at the positions in the subtraction, 1 is subtracted from 1, we must write $1 - 1 = 0$.

As an example, let us show how Johnny can balance 5 kg. First let us add $5 = 12_3$ and 1111_3. The result will be 1200_3. Hence,

$$5 = 1200_3 - 1111_3 = 0 \cdot 27 + 1 \cdot 9 - 1 \cdot 3 - 1 \cdot 1 = 9 - 3 - 1.$$

In order to balance 5 kg, one can place the 1 kg and 3 kg weights on the plate containing that load, and the 9 kg weight, on the other plate.

11.50. Examples of such representations are described in Problems 11.48 and 11.49. Now let us explain how such representations work for an arbitrary natural number n. Let us express the number n in the ternary system and consider the number $11\ldots11_3$ which is greater or equal to n. Let us add the numbers $11\ldots11_3$ and n in the ternary system. Their sum s has as many ternary digits as the number $11\ldots11_3$, and the first ternary digit of s is 1 or 2. Let us represent the number n as the difference $s - 11\ldots11_3$. In each position, the difference between the ternary digits is 1, 0, or -1. Let us construct two numbers in the ternary system using those ternary digits. The first one will consist of the 1's that we will place at the positions of positive differences, and zeros at the other positions; the second one will consist of 1's that we will place at the positions of negative differences, and zeros at the other positions. The number n is equal to the difference of these numbers.

11.51. The number $11111_3 = 121$ is greater than 100, hence any number between 1 and 100 can be represented as the difference of two numbers whose ternary expression contains only zeros and ones and has no more than 5 ternary digits (see the solution of Problem 11.50). One of these numbers corresponds to the weights on one plate, the other, to the weights on the other plate. Hence it suffices to have 5 weights of mass 1, 3, 9, 27, 81. Four weights are not enough, because using them one cannot weigh more than $3^4 - 1 = 80$ different weights (each weight is either in the left plate, or in the right plate, or does not participate in the weighing).

11.52. Suppose that the base of the number system used by Anton is a. Then what he wrote in his notebook means the following: $(a+3)^2 = a^2 + 7a + 1$. Solving this equation, one obtains $a = 8$. Since the digits 1, 3, and 7 used by Anton are less than this potential base, this is the correct answer.

11.53. If all the signs are pluses, the result is 127. Any odd integer m, $-127 \le m < 127$, is of the form $127 - 2k$, where $k \le 127$ is a natural number. To obtain $127 - 2k$, express k in the binary system as a 7-digit number (with leading zeroes if required) and in the expression

$$*1 * 2 * 3 * 4 * 5 * 6 * 7$$

replace by minuses the stars preceding the powers of 2 corresponding to 1's in the binary representation of k; the remaining stars should be replaced by pluses. For example, to represent the number $101 = 127 - 2 \cdot 13$ we obtain the binary representation $13 = 0001101_2$, so

$$101 = -1 + 2 - 4 - 8 + 16 + 32 + 64.$$

11.54. The solution of Problem 11.50 shows that the integers from 1 to $11111_3 = 121$ can be obtained in this way. Replacing pluses by minuses and minuses by pluses in each such expression, one can obtain the integers from -1 to -121.

11.55. Assume that such a table exists, and let us calculate the sum of all its numbers. On one hand, the table contains 5 rows, and the sum of numbers in each row equals 30. Hence the sum is 150. On the other hand, the table has 10 columns, and the sum of numbers of each of the columns is 10. Hence the sum is 100, a contradiction.

11.56. Let us find the sum of all the numbers if the table in two ways. The table has m rows, the sum of numbers in each row equals 1, hence the whole sum is m. Similarly, counting the sum of all the numbers via the columns, we obtain n as the result.

11.57. Half of the given numbers have the remainder 1 under division by 4, the other half, the remainder 3. Now let us paint the table as a chessboard and fill the black cells with the numbers having remainder 1 and the white cells with the numbers having remainder 3. One of many possible configurations is shown in the figure.

1	11	21	31
51	41	71	61
81	91	101	111
131	121	151	141

11.58. A number is divisible by 3 if and only if the sum of its digits is divisible by 3. To be definite, assume that the number not divisible by 3 is in the upper row. Then the sum of the numbers in each column is divisible by 3. Hence the total sum of numbers in the table is divisible by 3. From that sum let us subtract the sum of digits of the four numbers standing in the rows two to five. The result is divisible by 3, because all the subtrahends are divisible by 3. This is a contradiction.

11.59. If this is possible, then the values of the $2n+2$ sums (n rows, n columns, and 2 diagonals) will be different. Each of these sums consists of n summands that take one of the values $-1, 0, 1$. Hence each of these sums is an integer between $-n$ and n, so there are $2n+1$ such integers. Since $2n+1 < 2n+2$, there is a pair of such sums that take equal values. We arrived at a contradiction.

11.60. Suppose that Anna succeeded in placing the digits. Then each digit will occur in the table no more than 4 times. Indeed, if among the four lines where it occurs, there are two rows and two columns, then at their intersections there are exactly four cells for this digit (the digit can occur in two, or three, or in all four of these cells), whereas if there are three lines in one direction and one line in the other direction, then there are only three cells for the digit. However, there are only 10 digits, while there are 40 cells, hence there are exactly 4 digits of each type, and they are placed exactly at the intersection of two rows and two columns. In particular, in each column identical digits occur in pairs. But this is impossible since there are 5 digits (an odd number) in each column.

11.61. First let us prove that there are at least 1200 soldiers. Suppose that the number of files is m and the number of ranks is n. Then there are mn soldiers in the regiment and $\frac{mn}{100}$ were given new uniform. It follows from what we know that there is at least one soldier in a new uniform in at least $\frac{40n}{100}$ ranks, hence $\frac{mn}{100} \geq \frac{40n}{100}$, i.e., $m \geq 40$. Similarly, $\frac{mn}{100} \geq \frac{30n}{100}$, i.e., $n \geq 30$. Therefore, there are at least $40 \cdot 30 = 1200$ soldiers.

Now let us show that that there can be 1200 soldiers in the regiment. Let us line up the soldiers in a 40 by 30 rectangle, placing the 12 soldiers in new uniform along the diagonal. Clearly, there are soldiers in new uniform in exactly 30 % of the files and 40 % of the ranks.

11.62. The number of games played is equal to the number of defeated teams.

11.63. Assume that such a tournament can be organized. Then there will be $\frac{9 \cdot 3}{2}$ matches, which is nonsense.

11.64. One can place representatives of the teams (one for each team) in a circle. Now let two teams play with each other if and only if there are at least two people between their representatives (in each direction).

11.65. Ties never happen in volleyball, hence no two teams could both fail to win (they played with each other and one of them won). Therefore, exactly one team lost all the matches, and this constitutes 20 % of all the teams. So there were five teams.

11.66. For each participant, the total number of their wins equals the sum of the number of their wins with White and the number of their wins with Black. Since this player won as many games as all the others won playing with Black, the number of the wins of the player under consideration is equal to the total number of games in the tournament in which Black won, i.e., it is the same for all the participants.

11.67. In each game 2 points are in play. A total of $\frac{16 \cdot 15}{2} = 120$ games was played in the tournament, hence 240 points were in play. The teams that took the last nine places played a total of $\frac{9 \cdot 8}{2} = 36$ games, i.e., for them 72 points were put in play. Therefore, for the seven leading teams, there remained $240 - 72 = 168$ points, and the team that took sixth place earned no less than 22 points, the one that took fifth place, no less than 23 points, and so on. Since $21 + 22 + \ldots + 27 = 168$, it follows that each of the teams earned that number of points, in particular, the winning team earned 27 points. This number is odd, hence the winning team played to a tie at least once.

11.68. Suppose each played 6 games with each, A and B played all 6 games to a draw, B and C when playing with each other, won 3 games each, and A, when playing with C, won twice, lost once, and made 3 draws. Then, on one hand, A earned 6.5 points, B earned 6, while C earned 5.5. On the other hand, C won 4 games, B won 3, and A won 2.

11.69. Suppose that n players participated in the tournament. Then a total of $n(n-1)$ points were put in play. Hence each player scored $n-1$ points. Each player played $n-1$ games with White, and the number of games that he or she won with White is equal to one of the following n numbers: $0, \ldots, n-1$. Suppose that all the players obtained a different number of wins with White. Then all the possible variants from 0 to $n-1$ have actually occurred. Consider two participants A, who won $n-1$ gams with White, and B, who did not win any games playing the white pieces. What could be the result of the game that A played with Black against B? On one hand, A scored all their points playing with White, so he/she lost all their games played with Black, including the one against B. But B didn't win any games with White, so they couldn't have won this one. We arrived at a contradiction.

11.70. No player could have missed two consecutive games. Applying this to Kostya, who has played 8 games, one sees that at most $8 \cdot 2 + 1 = 17$ games have been played in all. Since Misha has played 17 games, the total number of games was precisely 17, so Misha participated in all and won at least the first 16 games. Thus, Kostya played all the games with even number, while Anton, all the games with odd numbers. Therefore, in the fifth game, Anton lost to Misha.

11.71. In the solution of Problem 11.31 it was proved that in any group of people there are two people who have the same number of friends in that group. This is the same problem (if we consider that teams that have already played become "friends" from that moment on).

11.72. Let us divide the giraffes into five groups with 5 giraffes in each. First order the heights of giraffes in each group. For this we will need five rounds. In the sixth round we compare the tallest giraffes in each of the five groups. After that, we denote the groups by the letters A, B, C, D, E in the descending order of the height of the tallest giraffe in the groups. Within each group the individual giraffes are distinguished by the indices 1, 2, 3, 4, 5, also in the descending order of the height of individual giraffes.

A_1	B_1	C_1	D_1	E_1
A_2	B_2	C_2	D_2	E_2
A_3	B_3	C_3	D_3	E_3
A_4	B_4	C_4	D_4	E_5
A_5	B_5	C_5	D_5	E_5

Note that the giraffes from the groups D and E cannot be prizewinners, because their height is less than that of any one of the giraffes A_1, B_1, or C_1. The giraffes with index 4 or 5 cannot be prizewinners either. Besides, $A_1 > B_1 > C_1 > C_2 > C_3$, so giraffes C_2 and C_3 cannot be prizewinners. And since $A_1 > B_1 > B_2 > B_3$, it follows that neither can B_3. Thus only the six giraffes can be prizewinners: A_1, A_2, A_3, B_1, B_2, C_1. But we already know that the giraffe A_1 is the tallest. In the seventh round, we compare the remaining five giraffes, obtaining those who win the 2nd and 3rd prizes.

11.73. In a tournament with 7 participants 21 games are played, with 8 participants, 28 games. More than 21 games have been played, so there are more than 7 participants. Less than 28 games have been played, so less than 8 participants have finished the tournament, and the total number of participants is less than 10. Hence either 8 or 9 participants played. For such numbers of participants, the number of games not played would be $28 - 23 = 5$ and $36 - 23 = 13$. Both of these numbers are odd. Now if Eugene and Alexander had played against each other, there would have been an even number of unplayed games: the same number for Eugene and Alexander.

11.74. The chess players who took the last 4 places played 6 games against each other and scored 6 points in these games. Hence the player who took the second place could not have scored less than 6 points. But this player could not have scored more that 6 points either. Indeed, if they scored 6.5 points, then the winner scored 7 points, i.e., the winner won all their games; but then the player who took the second place lost at least one game and could not have scored more than 6 points. Therefore, the player at the second place scored exactly 6 points and the last 6 players gained their points only in games with each other, i.e., each of them lost all the games to the first four. In particular, the 7th lost to the 3rd.

11.75. Each player played 4 games, a total of 10 games were played, and 10 points put in play. The winner scored at most 3 points, since he or she lost one game. But the winner could not have scored less than 3 points, because in that case the total number of points would be at most $2.5 + 2 + 1.5 + 1 + 0.5 = 7.5$. Hence the winner scored 3 points, winning 3 games and losing one to the second player since the latter is the only one who lost no game. The identity $3 + 2.5 + 2 + 1.5 + 1 = 10$ shows that the points could have been distributed only in that way, from 3 points for the first player to 1 point for the fifth one. The winner of the 2nd prize scored 1.5 points in the three other games and did not suffer any losses. Hence they played all these games to a draw. The player who took the 4th place scored 0.5 points in games with the winners of the first and second prizes, and 1.5 points in all. Had this player played all the other games to a draw, then they would not have won any game, but it's only the 5th player who won no game, so this is impossible. Had the 4th player won against the player who took 3rd place, the latter could not have scored more than 1.5 points, whereas he or she actually earned 2 points. Hence the 4th player lost to the 3rd player. Finally, the players who took the third and the fifth place played to a draw.

11.76. In the games of the 1st with the 2nd and with the 4th, there were no draws, hence the winner lost to the 2nd, but defeated the 4th. Thus the winner scored at most 3 points; arguing as in the solution of Problem 11.75, one concludes that the points scored in the tournament are 3, 2.5, 2, 1.5, and 1. Hence the winner defeated the 3rd, the 4th, and the 5th player, while the second drew their games. In the games with each other the 3rd, the 4th, and the 5th player earned 1.5, 1, and 0.5 points, respectively. The 4th player did not win any games, hence they obtained their score by playing 2 draws. Finally, the 3rd player beat the 5th one.

11.77. With 5 participants, this is possible (see the score sheet below).

	A	B	C	D	E
A	—	1	1/2	1/2	0
B	0	—	1	1/2	1/2
C	1/2	0	—	1	1/2
D	1/2	1/2	0	—	1
E	1	1/2	1/2	0	—

Now let us show that more than five players could not participate in such a tournament. Observe that the two following statements hold:

1) there is no triplet of players such that all of the games between them were drawn;

2) there is no triplet of players such that none of the games between them was drawn.

Assume that there are at least 6 players. Let's leave 6 of them and apply the result of Problem 11.32: consider that two players are friends if and only if they played to a draw. According to the indicated problem, there are either three players such that all of the games between them were drawn, or three players such that none of the games between them was drawn.

11.78. Let us prove that there could not be less than 6 teams. If there were, say, 5 teams, then among them they would have played $\frac{5 \cdot 4}{2} = 10$ matches and would have scored at least 20 points. Hence the only winning team scored more than $20 : 5 = 4$ points. But, according to the statement of the problem, that team earned no more than 5 points of the 12 possible ones. Therefore, the winning team scored exactly 5 points, and each of the other teams, no more than 4. Thus the total sum of points obtained by the teams would be at most $5+4\cdot 4 = 21$. But the number of points scored by the winning team implies that it won at least once and lost at least once, and in the case of two teams getting two wins the total sum of points obtained by all the teams cannot be less than 22, a contradiction. The arguments for such tournaments with two, or three, or four teams are similar. Below, two possible examples with six teams are presented.

Team	Points	1	2	3	4	5	6
1	**7**	—	1	1	1	1	3
2	**5**	1	—	1	1	1	1
3	**5**	1	1	—	1	1	1
4	**5**	1	1	1	—	1	1
5	**5**	1	1	1	1	—	1
6	**4**	0	1	1	1	1	—

Team	Points	1	2	3	4	5	6
1	**7**	—	3	3	0	0	1
2	**6**	0	—	1	1	3	1
3	**6**	0	1	—	3	1	1
4	**6**	3	1	0	—	1	1
5	**6**	3	0	1	1	—	1
6	**5**	1	1	1	1	1	—

11.79. Each team played 4 games. The first team tied once, and lost the other three games. The second team had 2 ties and 2 losses. Without wins, a team could have scored at most 4 points, hence the third team won once, tied twice, and lost once. Winning once and tying thrice, a team could have scored only 6 points, hence the fourth team had two wins, 1 tie, and 1 loss. As a result, the first 4 teams won 3 times and lost 7 times. The total number of wins equals the total number of losses, so the 5th team won all its games, scoring 12 points. It is not hard to give an example of such a tournament. Indeed, suppose the 5th team beat all the other teams, the 4th defeated the 1st and the 2nd, the 3rd defeated the 1st, and all the other games were ties. Then each of the teams will have the required number of points.

11.80. For example, the results of matches might have been as shown in the table below.

Team	1	2	3	4	Points
1	—	1	1	3	**5**
2	1	—	3	0	**4**
3	1	0	—	3	**4**
4	0	3	0	—	**3**

11.81. a) Suppose 7 participants won all their games against the other 5, and drew all the games among themselves. Then each one of them scored $5 + 0.5 \cdot 6 = 8$ points, which is more than 70 % of 11.

b) Suppose that 8 or more players obtained the title of chessmaster. Choose 8 of them. Each of them scored at least 8 points. Thus all together they scored at least 64 points. On the other hand, in the games with the other 4 participants, each of the chosen players scored at most 4 points. Thus, in the games among themselves the eight chosen participants scored at least 32 points all together. But they played only 28 games among themselves. This is a contradiction.

11.82. Let us rank the wrestlers according to their strength, giving them the ratings from 9 to 1. Then the sum of their ratings will be 45. Let us divide them into teams so that the sum of ratings of the teams should be equal. To do that, take the three rows of the magic square: $(2, 7, 6)$, $(9, 5, 1)$, and $(4, 3, 8)$. Then the score of the first team against the second one will be $5 : 4$, the second against the third, $5 : 4$, the third against the first, $5 : 4$.

11.83. Suppose that besides Johnny and Freddy, there were x participants. Then there were $x+2$ participants and $\frac{(x+2)(x+1)}{2}$ points were put in play. Each participant except Johnny and Freddy scored

$$\frac{1}{x} \cdot \Big(\frac{(x+2)(x+1)}{2} - 6.5\Big) = 0.5\Big(x + 3 - \frac{11}{x}\Big)$$

points. This number can be integer or half-integer only if $x = 11$.

11.84. If one chess player defeated another, let us draw an arrow from the winner to the loser. What we know about the lists means that for each player there is another one who can be reached by a sequence of 11 arrows, but cannot be reached by a sequence of less than 11 arrows. Hence, any player can be reached from any other player by a succession of arrows. Consider the following sequence of arrows: A_1 beat A_2; A_2 beat A_3; ...; A_{11} beat A_{12}. Then from A_2 all the players except A_1 can be reached by no more than 10 arrows. Hence A_i with i from 2 to 11 could not have defeated A_1 (otherwise it would be possible to reach all the others in less than 10 arrows). But someone defeated A_1 (otherwise it would be impossible to reach A_1), hence A_1 was defeated by A_{12}. As above, we can show that in the obtained cycle any player could have beaten only the next one. Therefore, only 12 games were not drawn, and there were $\frac{12 \cdot 11}{2} - 12 = 54$ draws.

11.85. If there are two players in the tournament, and one of them did not lose to the other, he or she can be assigned the number 1, and the other, the number 2. Suppose that we know how to number the participants of a tournament with n players, and suppose the number of participants of a tournament equals $n + 1$. Let us choose one of the participants and denote them by C. All the games involving the other players constitute a tournament with n participants.

We know that one can rank them $A_1, A_2, \ldots, A_n$ so that the player A_i did not lose to A_{i+1} (for all i from 1 to $n - 1$). Let m be the largest number for which the player C lost to the players $A_1, A_2, \ldots, A_m$ (if C did not lose to A_1, we put $m = 0$). Now the sequence A_1, A_2, ..., A_m, C, A_{m+1}, A_{m+2}, ..., A_n meets the requirements (if $m = 0$, then the sequence begins with C, while if $m = n$, it ends with C).

11.86. If one counts out 40 envelopes from a package of 100, then 60 will be left. The same with 90: it suffices to count out 10 from the package.

11.87. If we add the number of children who held hands with a boy to those who held hands with a girl, then we will obtain the number of all children plus the number of children who held hands with a girl and a boy. Hence there were $22 + 30 - 40 = 12$ children who held hands with a girl and a boy. Therefore, $30 - 12 = 18$ children held hands with girls only. These 18 children held $18 \cdot 2 = 36$ girl's hands, and 12 more held a girl's hand in one hand. Thus all the girls had $36 + 12 = 48$ hands, and so there were $48 : 2 = 24$ girls.

11.88. The first child from the left gave away 9 apples, hence he/she now had 9 apples less, the second child gave away 8 apples and was given 1, so he/she now had 7 apples less. Continuing that argument, we see that the first five children from the left now had 9, 7, 5, 3, and 1 apples less, respectively, while the next five had 1, 3, 5, 7, and 9 apples more, respectively. Each one of the last five children now had more apples, and in all they had $1+3+5+7+9 = 25$ apples more. The number of apples each of the other children had, decreased. Hence the number of apples in a group of children could have increased by 25 only if it was the group of the last five children. Hence the girls are these last five children.

11.89. From the identities

$$(12-1)^2 + (12+1)^2 = 2 \cdot 12^2 + 2 \cdot 1^2,$$
$$(12-2)^2 + (12+2)^2 = 2 \cdot 12^2 + 2 \cdot 2^2$$

it follows that

$$10^2 + 11^2 + 12^2 + 13^2 + 14^2 = 5 \cdot 12^2 + 2 \cdot (1^2 + 2^2) = 720 + 10 = 730.$$

11.90. In the identity

$$n^2 - (n-1)(n+1) = n^2 - (n^2 - 1) = 1,$$

put $n = 20\,202\,020$.

11.91. Immediate from the identity

$$n^2 + n^2(n+1)^2 + (n+1)^2 = (n^2 + n + 1)^2.$$

11.92. Immediate from the identity

$$(n-3)(n-1)(n+1)(n+3)+16=(n^2-5)^2.$$

11.93. There are exactly four equally likely outcomes: both coins come out heads; both coins come out tails; the first comes out heads, the second, tails; the first comes out tails, the second, heads. Petya wins in one of those cases, Vassya, in two.

11.94. Vassya can win only in one case, namely when both dice turn up with 6. Petya wins in two cases: when the first die comes up with 6 and the second one with 5, and when the first die comes up with 5 and the second one with 6.

11.95. The passengers who did not occupy their seats, if any, form a cycle which begins and ends with Freddy: each passenger occupies the seat of the next one, and the last one among those not seated at their place occupies Freddy's seat. Thus the last passenger could have occupied either his own seat or Freddy's. Those two seats are equivalent in the sense that neither the last passenger nor Freddy need to know what is written in their boarding passes: Freddy doesn't care and the last passenger will occupy the last free seat remaining, in any case; thus, if they exchange their boarding passes, nothing changes. Therefore, the last passenger will occupy one of these two seats with equal probability, and that probability is 1/2.

11.96. It is clear that there are at least 2 black balls in the box (otherwise we could never have taken out 2 black balls from it). All the balls cannot be black, otherwise we would always take out 2 black balls. So there are 2 or 3 black balls. If there were 2 black balls and 2 white ones, we would have also taken two white balls approximately 50 times, so we would have never (or hardly ever) taken one black and one white ball, which is highly unlikely. Hence there are, most likely, 3 black balls.

Let us see now how probable it is that we will take two black balls in approximately half the attempts. If we have 3 black balls and 1 white one, there are three equally likely ways to take 2 black balls, and there are three equally likely ways to take 1 black and 1 white ball. Hence if the take 2 balls randomly, then in approximately half of the attempts we will take 2 black balls.

11.97. For instance, $A=219$, $B=912$, and $A+B=1131$.

11.98. $101-10^2=1$.

11.99. The distance between the day when "yesterday" was "tomorrow", and the day when "the day after tomorrow" is "yesterday", is an odd number of days (namely 5). Hence these two days cannot be at the same distance from the same Sunday, but can only be at the same distance from different Sundays. This is possible if the first is Monday, the second is Saturday, and today is Wednesday.

11.100. All the families of the town may be divided into closed chains, in which each next family is followed by the family into whose house the previous family moved (it is possible that there is only one such chain). In chains consisting of an even number of families let us alternatively paint the houses in blue and green, then each family moves from a green house to a blue one or *vice versa*. In those chains where there is an odd number of families, let us paint one house in red and the remaining even number of houses in blue and green, alternating the blue and green colours. Then all the requirements will be met.

11.101. There are at most 90 apartments with two-digit numbers, hence there are at most 9 entrances. Consider two cases.

1. All the apartments with two-digit numbers are in Vassya's entrance. Then Vassya lives in the first entrance, and there are at least 99 apartments in it (otherwise there would be an apartment with a two-digit number in the second entrance). Hence there are at least 25 floors. Therefore, the house has 9 entrances, in each of which there can be 100, or 104, or 108 apartments ($112 \cdot 9 > 1000$). Hence there are 900, or 936, or 972 apartments in the house.

2. Not all the apartments with two-digit numbers are in Vassya's entrance. Denote the number of entrances by n; then there are $10n$ apartments with two-digit numbers in his entrance, where $1 \leq n \leq 8$. If that entrance is the first, then it contains apartments with n-digit numbers, hence there are $10n + 9$ apartments in Vassya's entrance, and their number is not divisible by 4. Therefore, in Vassya's entrance there cannot be 80, or 70, or 60, or 50 apartments with two-digit numbers (apartments with such numbers can only be in the first entrance). If there are apartments with three-digit numbers in Vassya's entrance, then the total number of apartments in all the previous entrances is $99 - 10n$, which is not divisible by 4 either. Thus, each apartment in Vassya's entrance has a two-digit number. If the number of these apartments is 40, then there are 4 entrances with 40 apartments in each. In that case, there are $40 \cdot 4 = 160$ apartments. If this number is less than 30, then there are no more than 3 entrances, and so there are no more than 90 apartments, which contradicts the condition that there are apartments with three-digit numbers in the house.

11.102. From the team of N men Chernomor will get at most $N-1$ coins, because the remainder is less than the divisor. Hence, in all he will get no more than $33-K$ coins, where K is the number of teams. However, if there is only one team, it follows that since $240 = 33 \cdot 7 + 9$, Chernomor will only get 9 coins. Thus Chernomor will not succeed in getting 32 coins.

a) Let us show how 31 coins can be obtained. For instance, Chernomor divides the men into two teams: 32 men in the first team and only one in the second, then gives 63 coins to the first team (of which 31 coins will be returned) and gives the remaining 177 coins to the only man in the second team.

b) In order to get 31 coins, Chernomor must divide the men into two teams and give 120 coins to each team. Then from the N-man team he must get $N-1$ coins. This means that 121 must be divisible by N. But 121 is only divisible by 1, 11, and 121, and it is impossible to obtain 33 by adding two of these numbers. So Chernomor will not succeed in getting 31 coins. However, he can get 30 coins. For instance, he can form one 27-man team and two 3-man teams and give 80 coins to each team. Since 81 is divisible by the number of men in each team, Chernomor will get $26 + 2 + 2 = 30$ coins.